Threshold subcontracting

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Abstract. *This paper presents an analytical approximation for threshold subcontracting in job-shop type production systems. Jobs arrive according to a Poisson process at a job shop with exponential service times and random routing. During a certain period (e.g., a week) the jobs are collected and at the end of the period it is decided how many (and which) jobs will enter the job shop at the beginning of the next period. I.e., when K is the threshold, there are $k \leq K$ jobs in the system and l jobs have arrived,* $\max\{k + l - K, 0\}$ *jobs are sent to a subcontractor. Numerical results show that throughput times are up to 40 percent lower than in open, i.e. unrestricted, job shops with the same machine utilization.*

Keywords: subcontracting, job shop, queueing network

1 Introduction

In job shop type production systems one strives for a high utilization but at the same time one wants to limit the throughput times of the job. In [1] Bertrand and Sridharan discuss subcontracting as an effective mechanism for balancing utilization and throughput time.

The system under consideration is a job-shop consisting of N stations with exponential service times and random routing. Jobs arrive to the job shop according to a Poisson process. During a certain period the newly arrived jobs are collected and at the end of that period it is decided how many of them are to be sent to a subcontractor. The rest of the jobs will enter the shop at the beginning of the next period. In the model it is assumed that all jobs are statistically identical. In practice one will have preferences about which jobs to accept depending on the requested due date, the client the job is for, the quality requirements, etc. By collecting the jobs during the period the subcontracting decisions can reflect these considerations, though only to a certain extent. It is assumed that the selection process at the end of the period has no information about the stations to be visited by the various jobs.

The simple subcontracting rule considered in this paper is completely characterized by a threshold. All jobs are allowed to enter as long as the total number of jobs in the shop does not exceed this threshold. If the threshold would be passed some jobs are sent to a subcontractor. Although the rule is very simple it is quite effective. It enables the decision maker to control the utilization of the workstations and the average time spent in the shop, although not independently.

Remark 1.1 *A much simpler system to analyse is the product form system in which the decision about acceptance is taken at the moment of arrival depending on the total number of jobs in the system. This however, makes it more difficult to control which jobs are accepted and which are sent to a subcontractor.*

This paper presents an analytical approximation for the performance of a job shop with job collection during a certain period and threshold subcontracting.

We will show that for any given threshold the two performance measures throughput time and utilization can be approximated from the analysis a one-dimensional Markov chain which gives an accurate approximation of the performance of the system.

In order to be able to do so, we have to approximate the departure process from the job shop during the period, if the period starts with a given number of jobs (less than or equal to the threshold). This will be done in Section 3. In Section 4 we derive the transition probabilities for the Markov chain. Section 5 introduces the relevant performance measures. In sections 6 and 7 we consider the symmetric and the non-symmetric shops.

But first we will describe the model in detail in Section 2.

2 The model

During the period jobs arrive to the job shop according to a Poisson process at rate λ. The arrivals are collected and at the end of the period it is decided how many jobs will enter the job shop at the beginning of the next period. The threshold for accepting jobs is denoted by K. If there are $k \leq K$ jobs in the system and l jobs have arrived then $\max\{k+l-K, 0\}$ jobs are sent to a subcontractor and are no longer considered in the system we study here. (In practice the selection of the jobs to send to a subcontractor is an important issue, but it is not considered in this model.) Once a job is accepted it will be completed.

The job shop consists of N single server stations. In station m the service times are exponential with mean b_m. If a job completes service in station m the job moves to station n with probability r_{mn} and leaves the job shop with probability $1 - \sum_n r_{mn}$.

Example 2.1 *In the numerical experiments we use as an example the case of a completely symmetric job shop with as period an 8 hour day, with $\lambda = 1$, $N = 5$, $b_m = 1$ and $r_{mn} = 4/25$, $n = 1, \cdots, 5$. (In practice one will see somewhat longer periods.) If all stations would have a utilization of 1, on the average 1 job per hour will leave the shop. As for any finite K the utilizations are less than 1 some jobs have to be sent to a subcontractor.*

3 The departure process

The departure process from the shop is complicated. At the beginning of the period a batch of jobs enters the system. During the period the number of jobs in the shop decreases with each departure, so the mean inter-departure times will increase. In order to model the departure process we first think of the job shop as of a closed queueing network in which each departing job is replaced by a new job which enters randomly in one of the stations. For that system the equilibrium distribution is easily determined. As all jobs are statistically identical, the state is fully described by the N-vector $(k_1, \cdots, k_N)$. From the product form distribution for closed networks we easily get the utilizations of the stations $u_m(k)$ depending on the population k(see e.g., Gordon and Newell [3]).

Example 3.1 *For the completely symmetric closed job shop with N stations and k jobs each state has the same equilibrium probability* $\binom{N+k-1}{k}^{-1}$ *And the utilization of each station is* $u_k = \frac{k}{N+k-1}$.

In order to determine the departure probabilities during the period we make the following assumption.

Steady state approximation. *If at the beginning of the period the job shop contains (including the newly admitted jobs) k jobs, then it is assumed that these jobs are distributed according to the equilibrium distribution of the closed network with k jobs .*

Based on this assumption, we approximate the departure process of the shop by a Poisson process with varying rate. At first the departure rate is $\mu_k = \sum u_m(k)(1-\sum r_{mn})/b_m$. After the first departure the Arrival Theorem for Closed Queueing Networks (which goes back to Reiser[3]) says that the remaining $k-1$ jobs are distributed according to the equilibrium distribution of the closed network with $k-1$ jobs. So after the first departure the new departure rate becomes μ_{k-1}, etc.

These inter-departure times are not completely independent, but the assumption that they are independent will be sufficiently close to reality.

We have to compute the departure process during a fixed time period, the time during which the jobs are collected.

For this we need more notation. Let $p_l^{(k)}(t)$ denote the probability of having l departures during a time t if the system starts with k jobs.

For $l = 0$ and $l = 1$ these probabilities are easily found. We get

$$p_0^{(k)}(t) = e^{-\mu_k t}, \tag{1}$$

and

$$p_1^{(k)}(t) = \int_0^t \mu_k e^{-\mu_k x} e^{-\mu_{k-1}(t-x)} dx = \frac{\mu_k}{\mu_k - \mu_{k-1}} \left(e^{-\mu_{k-1} t} - e^{-\mu_k t} \right). \tag{2}$$

For larger l it becomes more complicated. For $l = 2$ we get

$$p_2^{(k)}(t) = \frac{\mu_{k-1}}{\mu_{k-1} - \mu_{k-2}} \left[\frac{\mu_k}{\mu_k - \mu_{k-2}} \left(e^{-\mu_{k-2}t} - e^{-\mu_k t} \right) + \frac{\mu_k}{\mu_k - \mu_{k-1}} \left(e^{-\mu_{k-1}t} - e^{-\mu_k t} \right) \right]. \quad (3)$$

These expressions are not very transparent. Therefore it is better to exploit the recursive structure of these probabilities.

First note that the probabilities satisfy the recursive integral equation

$$p_l^{(k)}(t) = \int_0^t \mu_k e^{-\mu_k x} p_{l-1}^{(k-1)}(t-x)dx. \quad (4)$$

This suggests to write these probabilities in the form

$$p_l^{(k)}(t) = \sum_{j=k-l}^{k} A_{lj}^{(k)} e^{-\mu_j t} \;, l = 0, 1 \cdots, k-1. \quad (5)$$

From the integral equation one gets

$$p_l^{(k)}(t) = \int_0^t \mu_k e^{-\mu_k x} \sum_{j=k-l}^{k-1} A_{l-1,j}^{(k-1)} e^{-\mu_j (t-x)} dx \;. \quad (6)$$

Solving this gives a number of relations between the constants A.

$$A_{lj}^{(k)} = A_{l-1,j}^{(k-1)} \frac{\mu_k}{\mu_k - \mu_j} \;, \; j = k-l, \cdots, k-1 \quad (7)$$

and

$$A_{lk}^{(k)} = - \sum_{j=k-l}^{k-1} A_{l,j}^{(k)}. \quad (8)$$

The expression for the probability that the shop empties completely is obtained by subtracting the other probabilities from 1

$$p_k^{(k)}(t) = 1 - \sum_{l=0}^{k-1} p_l^{(k)}(t). \quad (9)$$

So given t and using the parameters μ_k from the closed shop we can compute all departure probabilities.

Unfortunately when using these expressions one encounters numerical difficulties because in the computations large positive and negative numbers of almost equal size have to be added.

In order to avoid these difficulties we switch to a uniformization approach (see e.g. Tijms[5, p. 154]). Here it works as follows. First one adds to each departure rate an artificial departure rate $\alpha - \mu_k$, with α equal to the largest value of the set of μ_k we are interested in. Next events are generated according to a Poisson process with rate α. Each such event is either a real event with probability μ_k/α or an artificial one. Finally, given the number of events, we recursively compute the distribution of the number of jobs still in the system. Define $r_i(l)$ to be the probability that there are i jobs in the system after l events, some of which are artificial, if the shop starts with k jobs. (So, in the notation we suppress the dependence on k.) Then

$$r_i(l) = r_i(l-1)\frac{\alpha - \mu_i}{\alpha} + r_{i+1}(l-1)\frac{\mu_{i+1}}{\alpha} \ , \ i = 0, 1, \cdots, k-1 \tag{10}$$

and

$$r_k(l) = r_k(l-1)\frac{\alpha - \mu_k}{\alpha} \ , \tag{11}$$

with initial distribution $r_k(0) = 1$ and $r_i(0) = 0$ for $i < k$.

The overall result from the departure process follows from the Poisson distribution of the number of events. One easily sees that this way the computation is stable as it involves adding nonnegative numbers only.

4 The transition probabilities

The arrival probabilities are much easier as they are just the Poisson probabilities. So the probability $q_k(t)$ of k arrivals in time t satisfies

$$q_k(t) = \frac{(\lambda t)^k}{k!} e^{\lambda t}, k = 0, \cdots. \tag{12}$$

For the Markov chain representing the shop with threshold K and period length t the transition probabilities $p_{kj}, \ k, j \le K$ satisfy

$$p_{kj} = \sum_{l=max(k-j,0)}^{k} p_l^{(k)}(t) q_{j-k+l}(t) \ , j = 0, \cdots, K-1 \tag{13}$$

and

$$p_{kK} = \sum_{l=0}^{k} p_l^{(k)}(t) \sum_{i=K-k+l}^{\infty} q_i(t) \ . \tag{14}$$

The equilibrium probabilities π_k follow from

$$\pi_k = \sum_{l=0}^{K} \pi_l p_{lk} \ , \tag{15}$$

together with the normalizing condition $\sum \pi_k = 1$.

The solution is determined numerically.

5 Performance measures

The following two important performance measures are easily obtained from the equilibrium distribution: the mean number of jobs in the system at the beginning of a period, L say, and the mean number of jobs sent to a subcontractor, R say.

$$L = \sum k\pi_k \tag{16}$$

and

$$R = \sum_k \pi_k \sum_{l=0}^{k} p_l^{(k)}(t) \sum_{i=K-k+l}^{\infty} q_i(t)(i-(K-k+l)). \tag{17}$$

The infinite sum can be rewritten as a finite sum using

$$\sum_{i=j}^{\infty}(i-j)q_i = \lambda t - j\sum_{i=0}^{j-1}(i-j)q_i. \tag{18}$$

From R we also get A, the mean number of accepted jobs per period, as $A = \lambda t - R$. Then the mean number of periods (to be more precise, starts of a period)for an accepted job in the system is equal to L/A.

A more detailed performance measure is the mean number of jobs in the system at an arbitrary point in time. Similarly as we computed the $p_l^{(k)}$ one may derive expressions for the mean number of jobs x time units after the beginning of a period with k jobs, and from that the mean number of jobs in the system.

However, there will be only a negligible difference between the detailed result from this computation and the result obtained from the intuitive (though not quite correct) reasoning that a departure takes place on the average half way the period. Using that we get a perfect approximation for S the mean time spent in the shop by an accepted job. With $L_S = L - A/2$ as approximation of the mean number of jobs in the system we get

$$S = tL_S = tL/A - t/2)\ . \tag{19}$$

Clearly S and A depend on K and the higher A the higher S. By choosing K we can control the overall performance.

6 Symmetric versus Non-symmetric

There is a crucial difference between a symmetric job shop and a non-symmetric shop. The difference lies in the behaviour of the utilizations. As we have seen (Example 3.1), in the symmetric job the utilizations increase only slowly with the total number of jobs in the system. In a non-symmetric shop there will be a single bottleneck station. For that station the utilization converges exponentially fast to 1. The conclusion is, that in a symmetric shop we need a large number of jobs to efficiently use the stations, whereas in a non-symmetric shop we expect that already a much smaller number suffices.

The analysis for the non-symmetric case is just as simple as for the symmetric case. For a closed network the utilizations can be determined e.g. by Buzen's algorithm [2]. From that the μ_k can be computed.

7 Two Examples

In this section we will consider an example for the symmetric shop and one for the non-symmetric one.

7.1 The symmetric shop

Let us consider as an example the case of an 8 hour day with $\lambda = 1$, $N = 5$, $b = 1$ and $1 - p = 1/5$.

For $K = 20$ the utilizations in the closed system are 20/24=0.83. Then the departure rate will be 0.83 as well, or 6.67 jobs per day. So at least 1 out of every 6 jobs will not be accepted.

Table 1 displays the results from the analysis of this system for various values of K. The first column gives K, the second one the throughput per day (at most 8), the third one the utilization of the stations, the fourth and the fifth one the approximations for the mean number of jobs in the shop and the mean number of days spent in the shop by an accepted job.

K	Thr	ρ	L_S	S
10	4.936	0.617	7.127	1.444
20	6.253	0.782	15.196	2.430
30	6.783	0.848	23.429	3.454
40	7.067	0.883	31.710	4.487
50	7.243	0.905	40.012	5.524
60	7.364	0.920	48.324	6.563
70	7.451	0.931	56.643	7.602
80	7.517	0.940	64.965	8.642
90	7.569	0.946	73.289	9.683
100	7.611	0.951	81.616	10.723

Table 1

7.2 The non-symmetric shop

The second example concerns a non-symmetric job shop. We again take $\lambda = 1$, $N = 5$ and $b = 1$. We will further take all parameters such that the utilization of station 5 is α times the utilization of the stations 1 to 4. For $\alpha > 1$ station 5 will be the bottleneck.

This is achieved if the jobs arrive in each of the stations 1 to 4 with probability $1/(4+\alpha)$ and in station 5 with probability $\alpha/(4+\alpha)$. After each service completion a job leaves the system with probability 1/5 and goes to any of the 5 (!) stations with a probability 4/5 times the probability of entering the shop in that state.

Note that the overall capacity of this shop is no longer 1 job per hour because the utilization of the stations 1 to 4 is equal to $1/\alpha$ times the utilization in station 5 and thus limited by $1/\alpha$. The capacity of the shop is thus $(4+\alpha)/5\alpha$.

Assuming that the shop with k jobs is in the closed network equilibrium, the departure rate is equal to 1/5 times the sum of the equilibrium utilizations. One may show that the bottleneck utilization met k jobs, denoted by $\rho_5(k)$ satisfies

$$\rho_5(k) = \frac{\sum_{l=1}^{k} \alpha^l \binom{k-l+3}{3}}{\sum_{l=0}^{k} \alpha^l \binom{k-l+3}{3}} \tag{20}$$

The utilizations in the other stations are a factor α smaller. So the capacity or output rate of the shop with k jobs satisfies

$$\mu_k = \rho_5(k) \frac{4+\alpha}{5\alpha} \tag{21}$$

It is not quite trivial how to compare the results for the symmetric and non-symmetric shop. We can take $\lambda = 1$ in both cases but there is a difference. For the symmetric shop this corresponds to the capacity for $K = \infty$, but for the non-symmetric shop the capacity with unrestricted acceptance is less than 8 per day. Therefore we also give the results in the non-symmetric case for the arrival rate equal to the limiting capacity: $(4+\alpha)/5\alpha$.

Tables 2 and 3 give the results for $\alpha = 1.2$. We have added the column ρ_{rel} which gives the relative utilization of the shop, which is of course just the utilization of the bottleneck station.

$\lambda = 1$ and $\alpha = 1.2$					
K	Thr	ρ	ρ_{rel}	L_S	S
10	4.911	0.614	0.708	7.146	1.455
20	6.163	0.770	0.889	15.342	2.490
30	6.610	0.826	0.953	23.930	3.620
40	6.800	0.850	0.981	32.893	4.837
50	6.882	0.860	0.993	42.227	6.136
60	6.915	0.864	0.997	51.859	7.499
70	6.927	0.866	0.999	61.682	8.904
80	6.931	0.866	1.000	71.607	10.331
90	6.933	0.867	1.000	81.579	11.767
100	6.933	0.867	1.000	91.568	13.207

Table 2

$\lambda = 0.867$ and $\alpha = 1.2$					
K	Thr	ρ	ρ_{rel}	L_S	S
10	4.809	0.601	0.694	6.876	1.430
20	5.987	0.748	0.863	13.958	2.332
30	6.409	0.801	0.924	20.526	3.203
40	6.600	0.825	0.952	26.591	4.029
50	6.700	0.837	0.966	32.282	4.818
60	6.757	0.845	0.975	37.727	5.584
70	6.792	0.849	0.980	43.021	6.334
80	6.816	0.852	0.983	48.224	7.075
90	6.833	0.854	0.986	53.370	7.811
100	6.846	0.856	0.987	58.480	8.543

Table 3

These results show that for the nonsymmetric shop a much lower threshold suffices to get the same relative utilization. In the open shop corresponding to $K = 30, \ \lambda = 1, \ \alpha = 1.2$, the utilization for the non-bottleneck station is 0.826 and for the bottleneck it is 0.953. This results in a system throughput time of 6 days. So the closed system with threshold gives a 40 percent reduction in throughput time. In the symmetric shop the difference is smaller.

8 Conclusions

We have presented an approximate analysis for a simple threshold strategy for subcontracting in a job shop. (Simulations show that the results are very accurate. There is hardly any difference between the results from the approximate analysis and the simulation.)

The analysis enables the decision maker to balance utilization and flow times. For the symmetric shop a high utilization has serious consequences for the flow times. In the non-symmetric shop however, high utilizations can be obtained with only a small increase in throughput time.

The threshold policy seems to be a simple but quite powerful instrument to control a job shop.

9 References

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