Minimum Variance Hedging for Managing Price Risks

Fikri Karaesmen
fkaraesmen@ku.edu.tr
Koç University

with Caner Canyakmaz and Süleyman Özekici

SMMSO Conference, June 4-9, 2017, Acaya - Lecce, Italy

My co-authors

Caner Canyakmaz

Prof. Süleyman Özekici

Outline

- Introduction and Literature
- The minimum variance approach
- A simple example for managing price risks
- Risk management in a newsvendor-like problem with Poisson demand and continuous prcie fluctuations
- A more complicated problem with multiple risks and dynamic hedging
- Numerical results

Risk sensitivity and management

- Capacity and inventory control decisions are usually taken to maximize an expected profit.
- But volatility of profit is a problem for risk-sensitive decision makers
- Operational risk hedging:
  - Hotels: Different customer segments (tourism and business)
  - Inventory management: Many products with different demand profiles, postponement of specifications etc.
- This talk: about risk management through variance minimization and financial hedging.

Literature Review

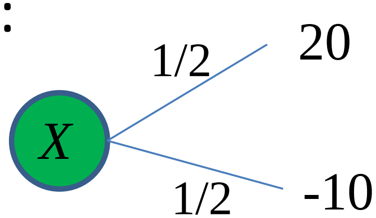
- Managing Risks in Inventory Management using Financial Hedging
  - Anvari (1987)
  - Gaur, Seshadri (2005)
  - Caldentey and Haugh (2006)
  - Chen, Simchi-Levi, Sun (2007)
  - Chod, Rudi, Van Mieghem (2010)
  - Kouvelis, Li, Ding (2013)
  - Kouvelis, Pang, Ding (2015)
  - Okyay, Karaesmen, Özekici (2015)
  - Sayın, Karaesmen, Özekici (2014)
  - Canyakmaz, Özekici, Karaesmen (2016)
  - Tanrisever (2017)

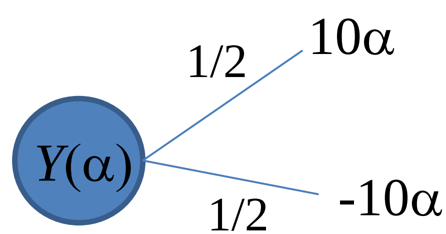
Hedging a risky operational project through variance minimization

- X: The returns from my ‘operation’ . We expect that $\mathrm{E}[X]>0$.

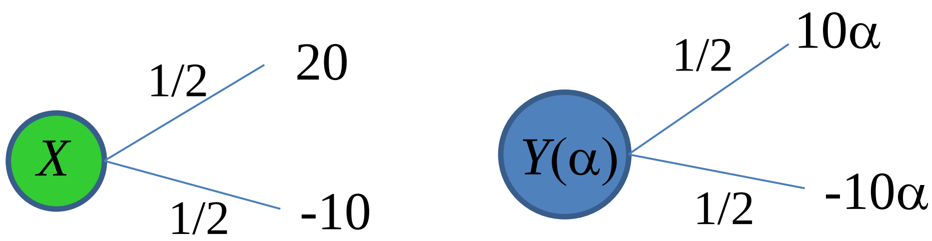
- $Y(\alpha)$: investment opportunity with returns proportional to investment level α, the total return for an investment level α is αY. Moreover, $\mathrm{E}[Y] = 0$.

- Example:





Hedging a risky operational project



$E[X+Y(\alpha)]=5$

$$Var[X+Y(\alpha)]=Var(X)+Var(\alpha Y)+2Cov(X,\alpha Y)$$
$$= Var(X)+\alpha^2 Var(Y)+2\alpha\, Cov(X,Y)$$

$$\alpha^* = \frac{-Cov(X,Y)}{Var(Y)} = -Corr(X,Y)\frac{\sigma_X}{\sigma_Y}$$

Hedging a risky operational project

- Furthermore, under the optimal level of investment:
- The reduction in variance is:

$$\Delta = \mathrm{Corr}(X,Y)^2\,\mathrm{Var}(X)$$

- And the relative reduction in variance (with respect to no investment) is :

  $\Delta_R = \Delta\,/\mathrm{Var}(X) = \mathrm{Corr}(X,Y)^2$

- A perfect hedge is possible when $\mathrm{Corr}(X,Y)$=1 (or =-1).
- We use market traded financial securities for 'Y'.
- The perfect hedge uses a combination of futures and options.
  - For a newsvendor problem, the perfect hedge uses a single future and a single option on Y (Gaur and Seshadri, 2005).

Hedging a risky operational project

- We can also consider multiple investments, Y_i, $i=1,2,..,n$:

$$\min_{\alpha_i} Var(X + \sum_{i=1}^{n} \alpha_i Y_i)$$

- And obtain: $$\boldsymbol{\alpha}^* = -\boldsymbol{C}^{-1}\boldsymbol{\mu}$$

where $\boldsymbol{C}$ is the variance - covariance matrix (of the random vector $\boldsymbol{Y}$) and $\boldsymbol{\mu}$ is the covariance vector of X with $\mathbf{Y}$, with $\mu_i=\mathrm{Cov}(X,Y_i)$.

Hedging Price Risks: a one-period discrete model

- We start with the simplest case: we are selling an item at T whose price P_T at T is random:
- If demand is not dependent on price:

$$\alpha^* = \frac{-Cov(P_T, Y)}{Var(Y)}$$

- And if demand at time is a function $g(P_T)$ of P_T :

$$\alpha^* = \frac{-Cov(g(P_T)P_T, Y)}{Var(Y)}$$

Hedging Price Risks: a model with Poisson demand arrivals and a continuous price process

- The prices fluctuate continously in $[0,T]$ according a stochastic price process
- Demand in $[0,T]$ is generated by a Poisson process whose rate at $\lambda(P_t)$ time t depends on the price P_t .

- Then:

$$\alpha^* = -\int_0^T \beta_u du \text{ where } \beta_u = \frac{Cov(P_u \lambda(P_u), Y)}{Var(Y)}.$$

- α* is an integrated ‘beta’ term.

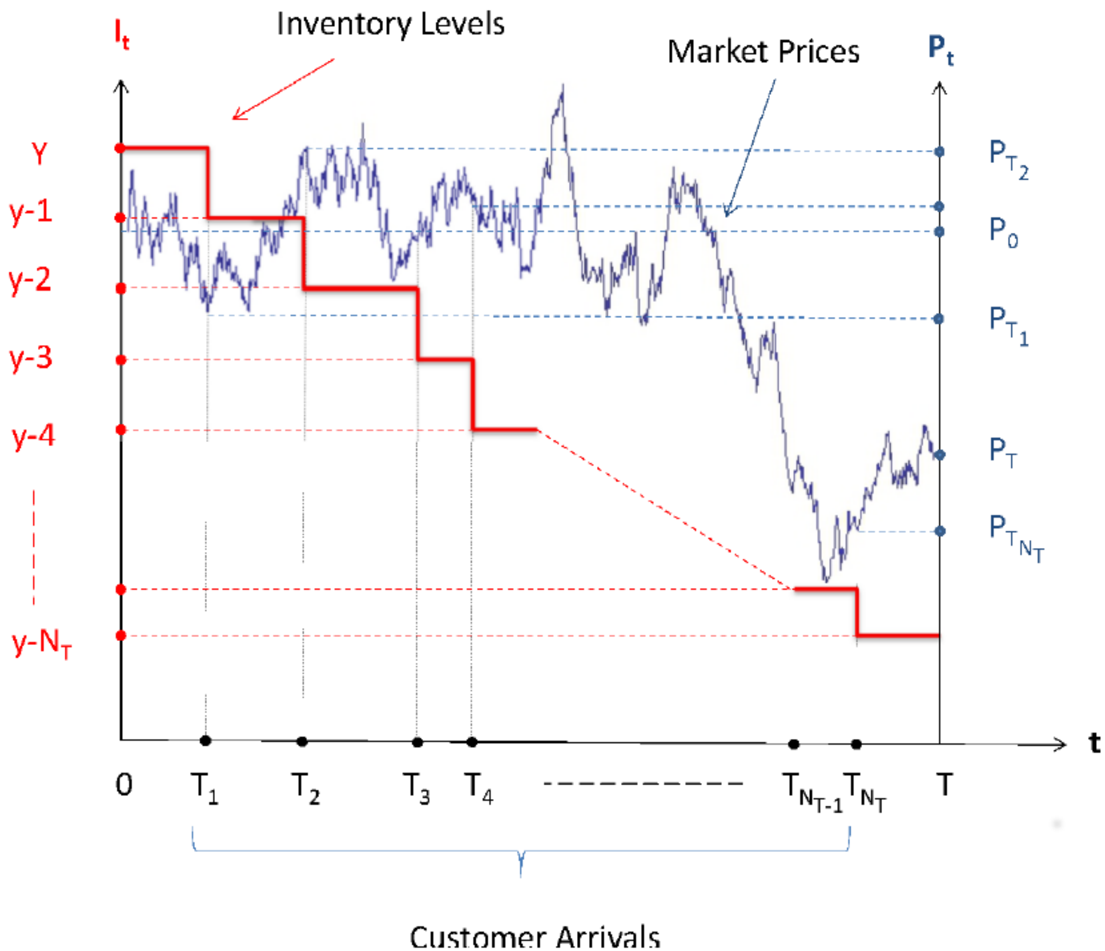
A newsvendor-like model with price risks and continuous fluctuations

- Assume that you have a starting inventory y that is to be sold in $[0,T]$.
- Unsold items at the end of the horizon cost h euros each and unsatisfied demand costs b euros each.
- Let N_t denote the total number of arrivals until time t.
- The total cashflow is:

$$CF = \sum_{j=1}^{N_T} P(T_j) - hE[(y - N_T)^+] - bE[(N_T - y)^+]$$

- There are now both inventory related and price related risks.

The Inventory Process with Price Fluctuations



Hedging with a Single Future

- Assume that S is a future on P_T. (this implies that S_0= P_0 and S_T= P_T) .

- Then the optimal hedge is:

$$\alpha^* = -\int_0^T \beta_t d_t + hCov((y - N_T)^+, S_T) + bCov((N_T - y)^+, S_T)$$

where

$$\beta_t = \frac{Cov(P_t \lambda(P_t), P_T)}{Var(P_T)}$$

Hedging with Multiple Assets

- Consider now multiple assets correlated with the price process: $S=\{S_1,S_2,..S_M\}$.

$$\alpha^*(y) = -C^{-1}\mu(y)$$

$$\mu_i(y) = \int_0^T Cov(P_u\lambda(u), S_i) - hCov\left((y-N_T)^+, S_i\right) - bCov\left((N_T-y)^+, S_i\right)$$

Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

We assume that there are M financial securities which are correlated with the market price process P

We let $S = \left(S^{(1)}, S^{(2)}, \ldots, S^{(M)}\right)$ denote the price processes for these securities where

$$S^{(i)} = \left\{S_t^{(i)}; t \geq 0\right\}$$

represents the price of the security i that is compounded to time T.

We assume that there are prespecified trading times

$\mathcal{T} = (t_0, t_1, t_2, \ldots, t_{n-1})$ with $t_0 = 0$ and $t_n = T$

We let $\theta_k = \left(\theta_k^{(1)}, \theta_k^{(2)}, \ldots, \theta_k^{(M)}\right)$ denote the portfolio decision at t_k where $\theta_k^{(i)}$ represents the financial position for ith security

Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

- The financial cashflow:

The final payoff of the financial portfolio at time T is

$$G(\theta, S) = \sum_{i=1}^{M} \sum_{k=0}^{n-1} \theta_k^{(i)} \left(S_{t_{k+1}}^{(i)} - S_{t_k}^{(i)} \right) = \sum_{k=0}^{n-1} (\theta_k)^T \widetilde{S}_{t_k}$$

$\widetilde{S}_{t_k}$ is an $M \times 1$ column vector that shows financial payoffs (compounded to time T) of holding a unit of each security during $[t_k, t_{k+1}]$

$\theta_k = \left(\theta_k^{(1)}, \ldots, \theta_k^{(M)} \right)$ is a column vector that represents the financial positions to hold at time t_k for securities $i = 1, ., M$

Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

For fixed y, we minimize the variance of the final cash flow which is

$$
\begin{aligned}
& Var\left(CF\left(y,N,\mathcal{P}\right)+\sum_{k=0}^{n-1}\theta_k\widetilde{S}_k\right) \\
= \quad & E\left[\left(CF\left(y,N,\mathcal{P}\right)+\sum_{k=0}^{n-1}\theta_k\widetilde{S}_k\right)^2\right]-\left\{E\left[CF\left(y,N,\mathcal{P}\right)\right]+E\left[\sum_{k=0}^{n-1}\theta_k\widetilde{S}_k\right]\right\}^2
\end{aligned}
$$

Since each $S^{(i)}$ is a martingale, the problem is equivalent to solving

$$
\min_{\theta} E\left[\left(CF\left(y,N,\mathcal{P}\right)+\sum_{k=0}^{n-1}\theta_k\widetilde{S}_k\right)^2\right]
$$

The objective function is separable in terms of dynamic programming

Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

At any time, we have four states: X, W, P, S

Evolution of inventory level

$$
\begin{aligned}
X_{k+1} &= X_k - N_{[t_k, t_{k+1}]} \\
X_0 &= y
\end{aligned}
$$

Evolution of wealth

$$
\begin{aligned}
W_{k+1} &= W_k + R_{[t_k, t_{k+1}]} + \theta_k \widetilde{S}_k \\
W_0 &= 0
\end{aligned}
$$

where operational revenue during t_k, t_{k+1} is

$$
R_{[t_k, t_{k+1}]} = \sum_{j=1}^{N_{[t_k, t_{k+1}]}} \alpha P_{Tj+t_k}
$$

Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

We can write the objective function as

$$E\left[\left(CF\left(y,\mathcal{N},\mathcal{P}\right)+\sum_{k=0}^{n-1}\theta_k\widetilde{S}_k\right)^2\right]=E\left[\left(w_n-\left[\left(b+P_{t_n}\right)\left(-x_n\right)^+ + hx_n^+\right]\right)^2\right].$$

We construct the dynamic programming formulation as follows:

$$\begin{aligned}V_k\left(x,w,p,s\right) \quad = \quad & \min_{\theta_k} E\left[V_{k+1}\left(x-N_{[t_k,t_{k+1}]},w+R_{[t_k,t_{k+1}]}+\theta_k\widetilde{S}_k,P_{t_{k+1}},S_{t_{k+1}}\right)\right.\\ & \left.|P_{t_k}=p,S_{t_k}=s\right]\end{aligned}$$

$$V_n\left(x,w,p,s\right)=\left(w-\left(b+p\right)\left(-x\right)^+ - hx^+\right)^2$$

Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

Theorem

(a) *Value function at any period k is of the form*

$$V_k(x,w,p,s) = g_k(x,w,p) + h_k(x,p,s)$$

where $g_k(x,w,p)$ *is given by*

$$g_k(x,w,p) = E\left[\left(w + R_{[t_k,t_n]} - (b + P_{t_n})\left(N_{[t_k,t_n]} - x\right)^+ - h\left(x - N_{[t_k,t_n]}\right)^+\right)^2 | P_{t_k} = p\right]$$

and $h_k(x,p,s)$ *is given by the following recursion*

$$\begin{aligned} h_k(x,p,s) &= -\mu_k(x,p,s)^T C_k(s)^{-1} \mu_k(x,p,s) \\ &\quad + E\left[h_{k+1}\left(x - N_{[t_k,t_{k+1}]}, P_{t_{k+1}}, S_{t_{k+1}}\right) \mid P_{t_k} = p, S_{t_k} = s\right] \end{aligned}$$

with the terminal condition

$$h_n(x,p,s) = 0.$$

Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

Theorem

(b) *The optimal portfolio at period k is given by*

$$\theta_k^*(x,p,s) = -C_k(s)^{-1}\mu_k(x,p,s)$$

where

$$(C_k(s))_{ij} = Cov\left(S_{t_{k+1}}^{(i)}, S_{t_{k+1}}^{(j)} | S_{t_k}^{(i)} = s^{(i)}, S_{t_k}^{(j)} = s^{(j)}\right)$$

and

$$\begin{aligned}(\mu_k(x,p,s))_j \quad = \quad & Cov\left(R_{[t_k,t_n]} - (b+P_{t_n})\left(N_{[t_k,t_n]} - x\right)^+ - h\left(x - N_{[t_k,t_n]}\right)^+,\right.\\ & \left. S_{t_{k+1}}^{(j)} \mid P_{t_k} = p, S_{t_k}^{(j)} = s^{(j)}\right).\end{aligned}$$

Summary

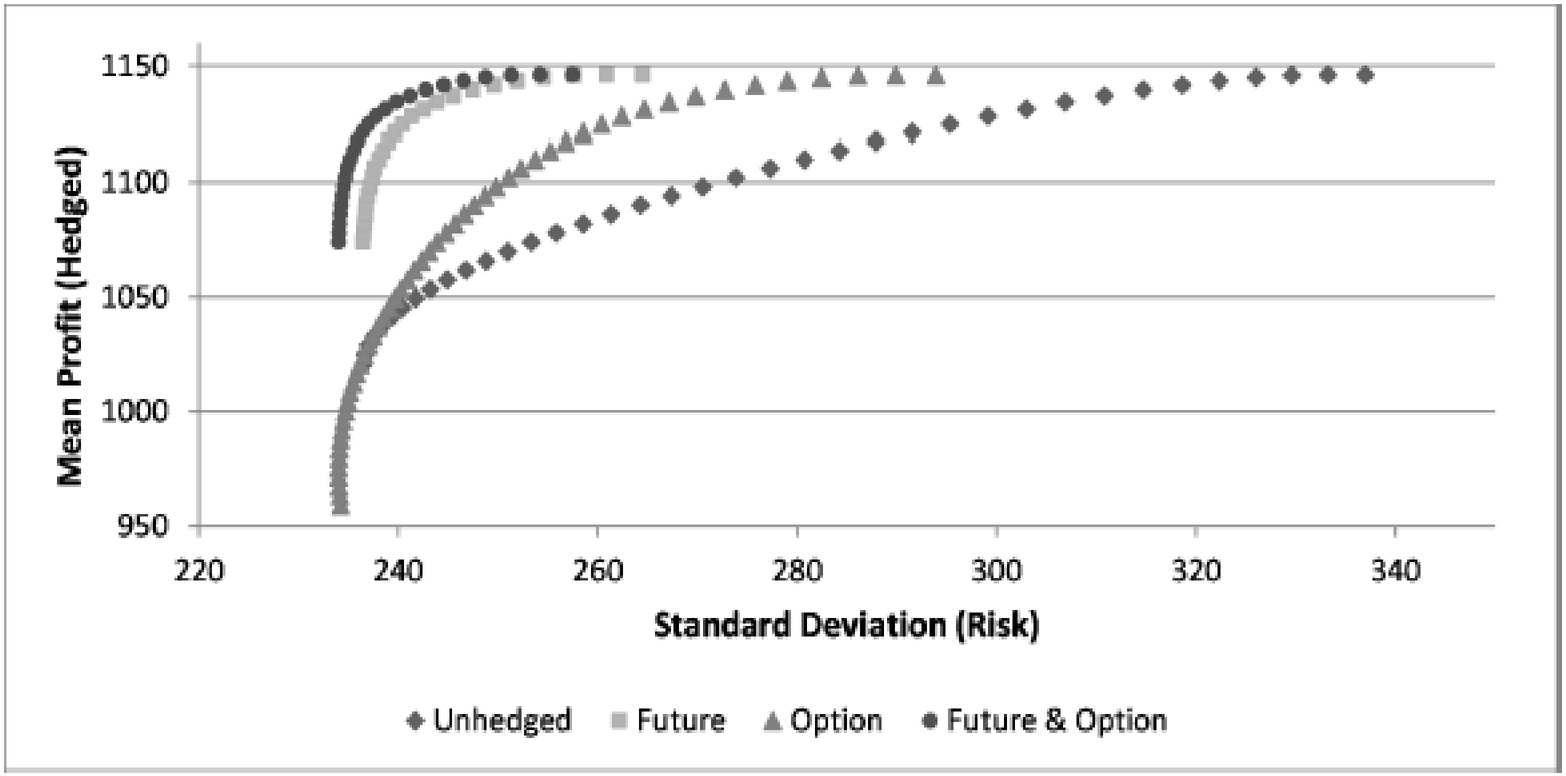
- We develop models for variance minimization of a risky operation (due to prices and demand) using a financial hedge.
- We can handle multiple assets, multiple trading points and multiple replenishments (not included today).
- We develop computational tools to obtain numerical solutions.
- This is a nice framework that leads to useful and insightful computational results.
  - Drawback: we are not performing a completely integrated optimization of operational and financial returns. The operational rules are fixed (so are the expected operational returns) and the hedge minimizes the variance.
  - But, we can easily relate this to the mean-variance framework.

Numerical Results

- We use futures and options (because their combinations lead to perfect hedges of fairly general operational cashflows for perfect correlation).
- We compare the following:
  - An unhedged operational cashflow
  - An optimally hedged operational cashflow using a single future
  - An optimally operational cashflow using a single option
  - An optimally operational cashflow using one future and one option

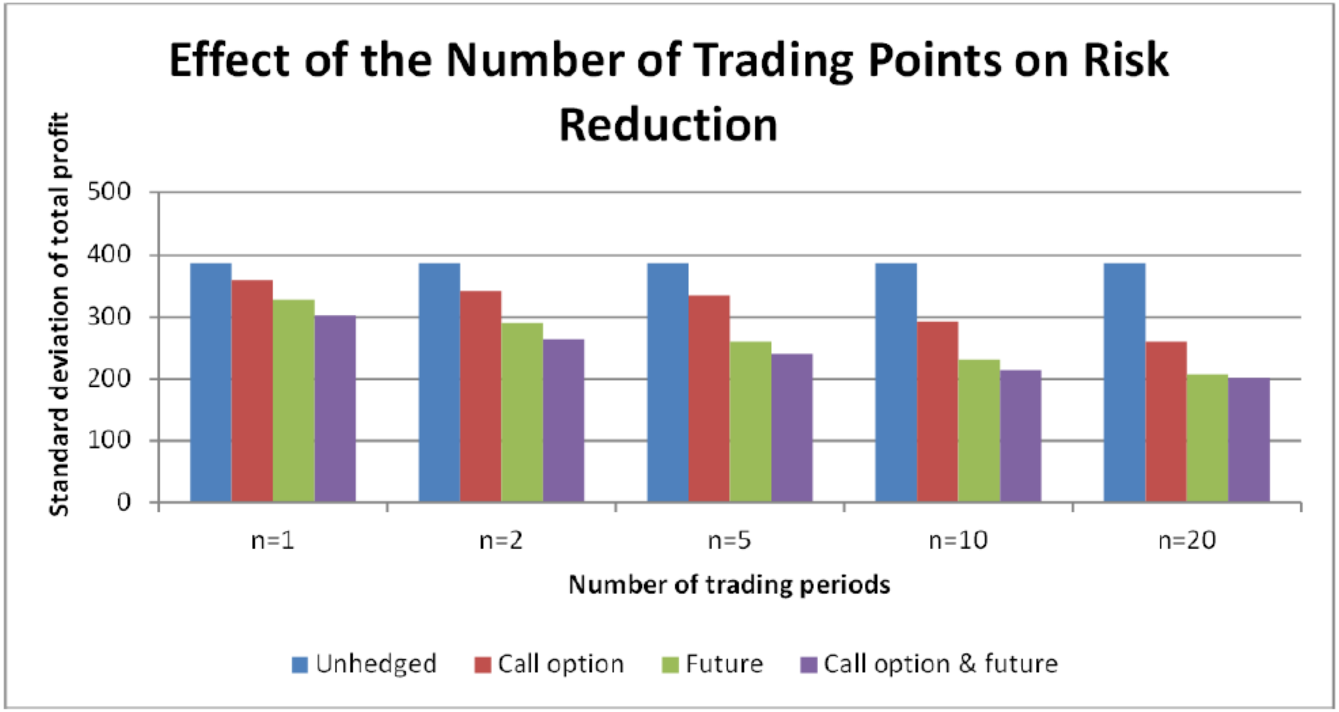
The Mean-Variance Efficient Frontier

By taking different inventory levels (order quantities), we can numerically trace the efficient frontier and let the decision maker choose.



The Effect of Dynamic Trading

- Mean of the cash flow$\sim$2000



For this example, choosing the right hedging portfolio has a more significant impact than increasing the frequency of trading.

Still to do

- Take into account parameter estimation risks
  - Robust optimization
  - Downside risk constraints
- Refinements
  - Budget constraints
  - Joint risk sensitivity
  - Investigating the nature of the hedging portfolio.
  - Making the empirical analysis work.

- Thank you for listening.
- Papers available at http://home.ku.edu.tr/~fkaraesmen/