Minimum Variance Hedging for Managing Price Risks

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Outline

• Introduction and Literature
• The minimum variance approach
• A simple example for managing price risks
• Risk management in a newsvendor-like problem with Poisson demand and continuous price fluctuations
• A more complicated problem with multiple risks and dynamic hedging
• Numerical results
Risk sensitivity and management

• Capacity and inventory control decisions are usually taken to maximize an expected profit.
• But volatility of profit is a problem for risk-sensitive decision makers
• Operational risk hedging:
  – Hotels: Different customer segments (tourism and business)
  – Inventory management: Many products with different demand profiles, postponement of specifications etc.
• This talk: about risk management through variance minimization and financial hedging.
Literature Review

• Managing Risks in Inventory Management using Financial Hedging
  – Anvari (1987)
  – Gaur, Seshadri (2005)
  – Caldentey and Haugh (2006)
  – Chod, Rudi, Van Mieghem (2010)
  – Kouvelis, Li, Ding (2013)
  – Kouvelis, Pang, Ding (2015)
  – Okyay, Karaesmen, Özekici (2015)
  – Sayın, Karaesmen, Özekici (2014)
  – Canyakmaz, Özekici, Karaesmen (2016)
  – Tanrisever (2017)
Hedging a risky operational project through variance minimization

- **X**: The returns from my ‘operation’. We expect that $\mathbb{E}[X] > 0$.

- **$Y(\alpha)$**: investment opportunity with returns proportional to investment level $\alpha$, the total return for an investment level $\alpha$ is $\alpha Y$. Moreover, $\mathbb{E}[Y] = 0$.

- Example:
Hedging a risky operational project

\[ E[X + Y(\alpha)] = 5 \]
\[ \text{Var}[X + Y(\alpha)] = \text{Var}(X) + \text{Var}(\alpha Y) + 2 \text{Cov}(X, \alpha Y) \\
= \text{Var}(X) + \alpha^2 \text{Var}(Y) + 2\alpha \text{Cov}(X, Y) \]

\[ \alpha^* = \frac{-\text{Cov}(X, Y)}{\text{Var}(Y)} = -\text{Corr}(X, Y) \frac{\sigma_X}{\sigma_Y} \]
Hedging a risky operational project

• Furthermore, under the optimal level of investment:
  
• The reduction in variance is:
  \[ \Delta = \text{Corr}(X,Y)^2 \text{Var}(X) \]

• And the relative reduction in variance (with respect to no investment) is :
  \[ \Delta_R = \Delta / \text{Var}(X) = \text{Corr}(X,Y)^2 \]

• A perfect hedge is possible when \( \text{Corr}(X,Y)=1 \) (or \(-1\)).
• We use market traded financial securities for ‘Y’.
• The perfect hedge uses a combination of futures and options.
  – For a newsvendor problem, the perfect hedge uses a single future and a single option on \( Y \) (Gaur and Seshadri, 2005).
Hedging a risky operational project

• We can also consider multiple investments, \( Y_i, i=1,2,..,n \):

\[
\min_{\alpha_i} \text{Var}(X + \sum_{i=1}^{n} \alpha_i Y_i)
\]

• And obtain: \( \alpha^* = -C^{-1}\mu \)

where \( C \) is the variance - covariance matrix (of the random vector \( Y \)) and \( \mu \) is the covariance vector of \( X \) with \( Y \), with \( \mu_i = \text{Cov}(X,Y_i) \).
Hedging Price Risks: a one-period discrete model

• We start with the simplest case: we are selling an item at \( T \) whose price \( P_T \) at \( T \) is random:

• If demand is not dependent on price:

\[
\alpha^* = \frac{-\text{Cov}(P_T, Y)}{\text{Var}(Y)}
\]

• And if demand at time is a function \( g(P_T) \) of \( P_T \):

\[
\alpha^* = \frac{-\text{Cov}(g(P_T)P_T, Y)}{\text{Var}(Y)}
\]
Hedging Price Risks: a model with Poisson demand arrivals and a continuous price process

- The prices fluctuate continuously in \([0, T]\) according to a stochastic price process.
- Demand in \([0, T]\) is generated by a Poisson process whose rate \(\lambda(P_t)\) at time \(t\) depends on the price \(P_t\).

Then:

\[
\alpha^* = -\int_0^T \beta_u du \quad \text{where} \quad \beta_u = \frac{\text{Cov}(P_u \lambda(P_u), Y)}{\text{Var}(Y)}.
\]

- \(\alpha^*\) is an integrated ‘beta’ term.
A newsvendor-like model with price risks and continuous fluctuations

• Assume that you have a starting inventory $y$ that is to be sold in $[0,T]$.
• Unsold items at the end of the horizon cost $h$ euros each and unsatisfied demand costs $b$ euros each.
• Let $N_t$ denote the total number of arrivals until time $t$.
• The total cashflow is:

$$CF = \sum_{j=1}^{N_T} P(T_j) - hE[(y - N_T)^+] - bE[(N_T - y)^+]$$

• There are now both inventory related and price related risks.
The Inventory Process with Price Fluctuations

Inventory Levels

Market Prices

$\mathcal{I}_t$

$y$

$y-1$

$y-2$

$y-3$

$y-4$

$y-N_T$

$t$

$0$, $T_1$, $T_2$, $T_3$, $T_4$, $T_{N-1}$, $T_N$, $T$

Customer Arrivals

$P_t$

$P_{T_2}$

$P_0$

$P_{T_1}$

$P_T$

$P_{T_{N_T}}$
Hedging with a Single Future

• Assume that $S$ is a future on $P_T$. (this implies that $S_0 = P_0$ and $S_T = P_T$).

• Then the optimal hedge is:

$$\alpha^* = -\int_0^T \beta_t d_t + hCov((y - N_T^+), S_T) + bCov((N_T - y)^+, S_T)$$

where

$$\beta_t = \frac{Cov(P_t \lambda(P_t), P_T)}{Var(P_T)}$$
Hedging with Multiple Assets

• Consider now multiple assets correlated with the price process: \( S = \{S_1, S_2, .., S_M\} \).

\[
\alpha^*(y) = -C^{-1}\mu(y)
\]

\[
\mu_i(y) = \int_0^T \text{Cov}(P_u \lambda(u), S_i) - h\text{Cov} \left((y - N_T)^+, S_i\right) - b\text{Cov} \left((N_T - y)^+, S_i\right)
\]
Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

We assume that there are $M$ financial securities which are correlated with the market price process $P$.

We let $S = \left( S^{(1)}, S^{(2)}, ..., S^{(M)} \right)$ denote the price processes for these securities where

$$S^{(i)} = \left\{ S^{(i)}_t ; t \geq 0 \right\}$$

represents the price of the security $i$ that is compounded to time $T$.

We assume that there are prespecified trading times

$T = (t_0, t_1, t_2, ..., t_{n-1})$ with $t_0 = 0$ and $t_n = T$

We let $\theta_k = \left( \theta^{(1)}_k, \theta^{(2)}_k, ..., \theta^{(M)}_k \right)$ denote the portfolio decision at $t_k$ where $\theta^{(i)}_k$ represents the financial position for $i$th security.
Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

• The financial cashflow:

The final payoff of the financial portfolio at time $T$ is

$$G (\theta, S) = \sum_{i=1}^{M} \sum_{k=0}^{n-1} \theta_k^{(i)} \left( S_{t_{k+1}}^{(i)} - S_{t_k}^{(i)} \right) = \sum_{k=0}^{n-1} (\theta_k)^T \tilde{S}_{t_k}$$

$\tilde{S}_{t_k}$ is an $M \times 1$ column vector that shows financial payoffs (compounded to time $T$) of holding a unit of each security during $[t_k, t_{k+1}]$

$\theta_k = \left( \theta_k^{(1)}, ..., \theta_k^{(M)} \right)$ is a column vector that represents the financial positions to hold at time $t_k$ for securities $i = 1, .., M$
Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

For fixed $y$, we minimize the variance of the final cash flow which is

$$\text{Var} \left( \text{CF} (y, N, \mathcal{P}) + \sum_{k=0}^{n-1} \theta_k \tilde{S}_k \right)$$

$$= \mathbb{E} \left[ \left( \text{CF} (y, N, \mathcal{P}) + \sum_{k=0}^{n-1} \theta_k \tilde{S}_k \right)^2 \right] - \left\{ \mathbb{E} \left[ \text{CF} (y, N, \mathcal{P}) \right] + \mathbb{E} \left[ \sum_{k=0}^{n-1} \theta_k \tilde{S}_k \right] \right\}^2$$

Since each $S^{(i)}$ is a martingale, the problem is equivalent to solving

$$\min_{\theta} \mathbb{E} \left[ \left( \text{CF} (y, N, \mathcal{P}) + \sum_{k=0}^{n-1} \theta_k \tilde{S}_k \right)^2 \right]$$

The objective function is separable in terms of dynamic programming.
Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

At any time, we have four states: $X, W, P, S$

Evolution of inventory level

$$X_{k+1} = X_k - N_{[t_k, t_{k+1}]}$$
$$X_0 = y$$

Evolution of wealth

$$W_{k+1} = W_k + R_{[t_k, t_{k+1}]} + \theta_k \tilde{S}_k$$
$$W_0 = 0$$

where operational revenue during $t_k, t_{k+1}$ is

$$R_{[t_k, t_{k+1}]} = \sum_{j=1}^{N_{[t_k, t_{k+1}]}} \alpha P_{T_j+t_k}$$
Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

We can write the objective function as

$$E \left[ \left( \frac{CF(y, N, P) + \sum_{k=0}^{n-1} \theta_k \tilde{S}_k}{\sum_{k=0}^{n-1} \theta_k \tilde{S}_k} \right)^2 \right] = E \left[ \left( w_n - \left( b + P_{t_n} \right) ( -x_n )^+ + hx_n^+ \right)^2 \right].$$

We construct the dynamic programming formulation as follows:

$$V_k(x, w, p, s) = \min_{\theta_k} E \left[ V_{k+1} \left( x - N_{[t_k, t_{k+1}]}, w + R_{[t_k, t_{k+1}]} + \theta_k \tilde{S}_k, P_{t_{k+1}}, S_{t_{k+1}} \right) \right]$$

$$\left| P_{t_k} = p, S_{t_k} = s \right|$$

$$V_n(x, w, p, s) = \left( w - ( b + p ) ( -x )^+ + hx^+ \right)^2$$
Hedging with Multiple Assets and Multiple Trading Times: A dynamic model

**Theorem**

(a) Value function at any period \( k \) is of the form

\[
V_k(x, w, p, s) = g_k(x, w, p) + h_k(x, p, s)
\]

where \( g_k(x, w, p) \) is given by

\[
g_k(x, w, p) = E \left[ \left( w + R_{[t_k, t_n]} - (b + P_{t_n}) \left( N_{[t_k, t_n]} - x \right)^+ - h \left( x - N_{[t_k, t_n]} \right)^+ \right)^2 \mid P_{t_k} = p \right]
\]

and \( h_k(x, p, s) \) is given by the following recursion

\[
h_k(x, p, s) = -\mu_k(x, p, s)^T C_k(s)^{-1} \mu_k(x, p, s)
\]

\[
+ E \left[ h_{k+1} \left( x - N_{[t_k, t_{k+1}]}, P_{t_{k+1}}, S_{t_{k+1}} \right) \mid P_{t_k} = p, S_{t_k} = s \right]
\]

with the terminal condition

\[
h_n(x, p, s) = 0.
\]
(b) The optimal portfolio at period $k$ is given by

$$
\theta_k^*(x, p, s) = -C_k(s)^{-1} \mu_k(x, p, s)
$$

where

$$(C_k(s))_{ij} = Cov\left(S_{t_{k+1}}^{(i)}, S_{t_{k+1}}^{(j)} | S_{t_k}^{(i)} = s^{(i)}, S_{t_k}^{(j)} = s^{(j)}\right)$$

and

$$(\mu_k(x, p, s))_j = Cov\left(R_{[t_k, t_n]} - (b + P_{t_n}) \left(N_{[t_k, t_n]} - x\right)^+ - h \left(x - N_{[t_k, t_n]}\right)^+, S_{t_{k+1}}^{(j)} | P_{t_k} = p, S_{t_k}^{(j)} = s^{(j)}\right).$$
Summary

• We develop models for variance minimization of a risky operation (due to prices and demand) using a financial hedge.
• We can handle multiple assets, multiple trading points and multiple replenishments (not included today).
• We develop computational tools to obtain numerical solutions.
• This is a nice framework that leads to useful and insightful computational results.
  – Drawback: we are not performing a completely integrated optimization of operational and financial returns. The operational rules are fixed (so are the expected operational returns) and the hedge minimizes the variance.
  – But, we can easily relate this to the mean-variance framework.
Numerical Results

• We use futures and options (because their combinations lead to perfect hedges of fairly general operational cashflows for perfect correlation).

• We compare the following:
  – An unhedged operational cashflow
  – An optimally hedged operational cashflow using a single future
  – An optimally operational cashflow using a single option
  – An optimally operational cashflow using one future and one option
The Mean-Variance Efficient Frontier

By taking different inventory levels (order quantities), we can numerically trace the efficient frontier and let the decision maker choose.
For this example, choosing the right hedging portfolio has a more significant impact than increasing the frequency of trading.
Still to do

• Take into account parameter estimation risks
  – Robust optimization
  – Downside risk constraints

• Refinements
  – Budget constraints
  – Joint risk sensitivity
  – Investigating the nature of the hedging portfolio.
  – Making the empirical analysis work.

• Thank you for listening.
  • Papers available at http://home.ku.edu.tr/~fkaraesmen/