Dynamic capacitated Lot Sizing with Random Demand and Random Yield

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Outline

1. The Problem
2. Model Formulation
3. Solution
4. Numerical Results
5. Conclusion
Problem definition

Dynamic Lotsizing, random demand, random yield

- A single resource with setups and limited capacity
- Multiple products with dynamic and random demand (normally distributed)

<table>
<thead>
<tr>
<th>Period $t$</th>
<th>6/2017</th>
<th>7/2017</th>
<th>...</th>
<th>12/2017</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_t$</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$\sigma_t$</td>
<td>...</td>
<td>...</td>
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</tbody>
</table>

- Random yield (Binomial yield with success probability $p$ per unit produced)
- ’Static-Dynamic Uncertainty’ policy (usually applied in a rolling planning environment)
- Service level: Fill rate per cycle ($\beta_c$)
Current industrial practice

- Forecast future demands and add a *scrap allowance*
- Use a lotsizing model (EOQ, SIULSP, CLSP, . . .).

**Result:** Uncontrollable safety stock and service level
Model SCLSP$^Y_{\beta c}$

Non-linear model formulation, I

$$\min Z = \sum_{k=1}^{K} \sum_{t=1}^{T} (s_k \cdot \gamma_{kt} + h_k \cdot E\{I_{kt}^{p,\text{end}}\})$$

s.t.

$$I_{k,t-1}^{n,\text{end}} + \underbrace{Q_{kt}(\hat{q}_{kt}, p_k)}_{\text{planned}} - D_{kt} = I_{kt}^{n,\text{end}}, \quad \forall k, \forall t$$

$$q_{kt} - M \cdot \gamma_{kt} \leq 0, \quad \forall k, \forall t$$

$$\sum_{k=1}^{K} (tb_k \cdot q_{kt} + tr_k \cdot \gamma_{kt}) \leq b_t, \quad \forall t$$
Net inventory $I^n$

Sequence of events

1. Net inventory at time $t-1$: $I_{t-1}^{n, prod}$
2. Demand at time $t-1$: $D_{k,t-1}$
3. Production at time $t$: $Q_{kt}$
4. Net inventory at time $t$: $I_t^{n, prod}$
5. Demand at time $t$: $D_{kt}$
6. Net inventory at time $t$: $I_t^{n, end}$
Model SCLSP$^{\gamma}_{\beta c}$

Non-linear model formulation, II

**Inventory on hand**

\[ I_{kt}^{p,\text{end}} = [I_{kt}^{n,\text{end}}]^+ \quad \forall k, \forall t \]

**Backlog**

\[ I_{kt}^{f,\text{prod}} = -[I_{kt}^{n,\text{end}} + Q_{kt}(q_{kt}, p_k)]^- \quad \forall k, \forall t \]

\[ I_{kt}^{f,\text{end}} = -[I_{kt}]^- \quad \forall k, \forall t \]

**Backorders**

\[ B_{kt} = I_{kt}^{f,\text{end}} - I_{kt}^{f,\text{prod}} \quad \forall k, \forall t \]
Model SCLSP$^Y_{\beta c}$

Non-linear model formulation, III

$$1 - \frac{E \left\{ \sum_{\tau = t - \ell_{kt}}^{t} B_{k \tau} \right\}}{E \left\{ \sum_{\tau = t - \ell_{kt}}^{t} D_{k \tau} \right\}} \geq \beta^c_k$$

$$\forall k, \forall t \in \{ t | \gamma_{k,t+1} = 1 \}$$

$$\ell_{kt} = (\ell_{k,t-1} + 1) \cdot (1 - \gamma_{kt})$$

$$\ell_{k0} = -1$$

$$\ell_{kt} \geq 0$$

$$q_{kt} \geq 0$$
Net inventory $I^n$

\[ I_{n,\text{prod}}^{t-1} = I_{n,\text{prod}}^{t-1} + Q_{kt}(q_k, p_k) - D_{k,t-1} \]

\[ I_{n,\text{end}}^{t-1} = I_{n,\text{end}}^{t-1} + Q_{kt}(q_k, p_k) - D_{kt} \]
The Problem
Model Formulation
Solution
Numerical Results
Conclusion

Single-Item Problem
Multiple-Item Problem (Finite Capacity)

Net inventory $I_{kt}^n$

Parameters (normal distribution)

Expected values

$$\mu_{I_{kt}^n, \text{prod}} = \mu_{I_{k,t-1}^n, \text{prod}} + \mu Q_{kt} - \mu D_{k,t-1}$$

$$\mu_{I_{kt}^n, \text{end}} = \mu_{I_{k,t-1}^n, \text{end}} + \mu Q_{kt} - \mu D_{kt}$$

Variances

$$\sigma^2_{I_{kt}^n, \text{prod}} = \sigma^2_{I_{k,t-1}^n, \text{prod}} + \sigma^2_{Q_{kt}} + \sigma^2_{D_{k,t-1}}$$

$$\sigma^2_{I_{kt}^n, \text{end}} = \sigma^2_{I_{k,t-1}^n, \text{end}} + \sigma^2_{Q_{kt}} + \sigma^2_{D_{kt}}$$
Performance criteria

Inventory on hand and backorders

**Inventory on hand** at the end of period \( t \)

\[
E\{I_{kt}^{p,\text{end}}\} = E\{\left[ I_{kt}^{n,\text{end}} \right]^+ \}
\]

**Backorders** in period \( t \)

\[
E\{B_t\} = E\{I_{kt}^{f,\text{end}}\} - E\{I_{kt}^{f,\text{prod}}\}
\]

**Backlog** at the end of period \( t \)

\[
E\{I_{kt}^{f,\text{end}}\} = - \left( E\{I_{kt}^{n,\text{end}}\} - E\{I_{kt}^{p,\text{end}}\} \right)
\]
Shortest-Path Representation

\[ E\{C_{12}\} \quad E\{C_{23}(P_2)\} \quad E\{C_{34}(P_3)\} \]

\[ E\{C_{13}\} \quad E\{C_{14}\} \quad E\{C_{24}(P_2)\} \]

Single-Item Problem
Multiple-Item Problem (Finite Capacity)
Shortest-Path Representation (3 periods)
Production in periods 1 and 2
Evaluation of an edge from $\tau$ to $t$

\[ E\{C_{\tau t}\} = s + h \cdot \sum_{\ell=\tau}^{t-1} \left[ I_{\tau-1}^{P,\text{end}}(P) + Q[q^*(P_{\tau}), p_k] - \sum_{i=\tau}^{\ell} D_i \right] + \]

$P_{\tau}$ – Path from 1 to $\tau$

Initial inventory at the beginning of period $\tau$ depends on path $P_{\tau}$

$q^*_{\tau t}$ – planned lot size required to meet target service level $\beta_c$
Multi-item problem

Solution approaches

- $ABC^Y_\beta$ heuristic
- Column generation (set partitioning model)
- MIP with piecewise-linear approximation of inventory and backorder functions
Dynamic and random demand with finite capacities

ABC$^\gamma$ Heuristic

**Basic Principle**

Transform the matrix of demands into a matrix of production quantities

\[
\begin{align*}
  d_{11} & \rightarrow d_{12} \rightarrow d_{13} \rightarrow \cdots \\
  d_{21} & \rightarrow d_{22} \rightarrow d_{23} \rightarrow \cdots \\
  d_{31} & \rightarrow d_{32} \rightarrow d_{33} \rightarrow \cdots \\
  d_{41} & \rightarrow d_{42} \rightarrow d_{43} \rightarrow \cdots \\
\end{align*}
\]

\[
\begin{align*}
  q_{11} & \quad - \quad - \quad \cdots \quad \text{look ahead} \\
  q_{21} & \quad - \quad q_{23} \quad \cdots \quad T \\
  q_{31} & \quad q_{32} \quad q_{33} \quad \cdots \\
  q_{41} & \quad q_{42} \quad - \quad \cdots \\
\end{align*}
\]
Dynamic and Random Demand with Finite Capacities

$ABC_\beta$ Heuristic

1. procedure $ABC_\beta$
2. for $\tau = 1, 2, \ldots T$ do
3. Call $\text{CREATELOTS}(\tau)$
4. if $\left( \sum_{k=1}^{K} (tp_k \cdot Q[q_{kt}, p_k] + ts_k \cdot \gamma_{kt}) < b_\tau \right)$ then
5. Call $\text{EXTENDLOTS}(\tau)$
6. else
7. Call $\text{SHIFTPRODUCTION}(\tau)$
8. end if
9. end for
10. end procedure
Column Generation Heuristic
Generation of Production Plans ($\beta_c$ service level)

For each product $k$ create a set of alternative production plans and select exactly one plan.

<table>
<thead>
<tr>
<th>$c_{kn}$</th>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$n$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>3487.18</td>
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<td>2807.74</td>
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<td>$t$</td>
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<td>132.46</td>
<td>641.64</td>
<td>326.39</td>
<td>132.46</td>
<td>28.40</td>
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<td>106.65</td>
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<td>386.39</td>
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<td>203.52</td>
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<td>111.73</td>
<td>4</td>
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<td>–</td>
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<td>228.11</td>
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</tr>
<tr>
<td>132.29</td>
<td>5</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>167.85</td>
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<tr>
<td>118.47</td>
<td>6</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>
Column Generation Heuristic
Set Partitioning Model: SCLSP\textsubscript{SPP}

Minimize \( Z = \sum_{k=1}^{K} \sum_{n=1}^{P_k} c_{kn} \cdot \delta_{kn} \)

subject to

\[
\sum_{k=1}^{K} \sum_{n=1}^{P_k} \kappa_{knt} \cdot \delta_{kn} \leq b_t \quad t = 1, 2, \ldots, T \quad (\pi_t)
\]

\[
\sum_{n=1}^{P_k} \delta_{kn} = 1 
\]

\[
\delta_{kn} = \{0, 1\} 
\]

\[
k = 1, 2, \ldots, K; \quad n = 1, 2, \ldots, P_k
\]
Column Generation Heuristic

Solution Procedure

Basic Structure of the CG Heuristic

1. Generate production plans (e.g. with the $\text{ABC}_\beta$ heuristic)
2. Apply standard column generation procedure
3. For the remaining items: apply $\text{ABC}_\beta$ heuristic
MIP based Heuristic

Piecewise Linear Approximation of Inventory on hand

\[
E \{ I_t^p \}
\]
MIP based Heuristic

Piecewise Linear Approximation of Backorders
MIP based Heuristic

Piecewise Linear Approximation of Inventory on hand

Slope of the *inventory on hand* function for line segment $\ell$:

$$\Delta_{I_{\text{p, end}}}^\ell = \frac{E \left\{ I_{kt}^{\text{p, end}}(u_{kt}^\ell) \right\} - E \left\{ I_{kt}^{\text{p, end}}(u_{kt}^{\ell-1}) \right\}}{u_{kt}^\ell - u_{kt}^{\ell-1}}$$

Slope of the function of the *backorders* for line segment $\ell$ is

$$\Delta_{B_{kt}}^\ell = \frac{E \left\{ B_{kt}(u_{kt}^\ell) \right\} - E \left\{ B_{kt}(u_{kt}^{\ell-1}) \right\}}{u_{kt}^\ell - u_{kt}^{\ell-1}}$$
MIP based Heuristic
Piecewise Linear Approximation; Multi-item

Minimize \( E\{C\} = \sum_{k=1}^{K} \sum_{t=1}^{T} (s_k \cdot \gamma_{kt} + h \cdot \left[ \Delta_{ip}^{0} + \sum_{\ell=1}^{L} \Delta_{ip}^{\ell} \cdot w_{kt}^{\ell} \right]) \)

subject to

\[ \sum_{k=1}^{K} (tb_k \cdot q_{kt} + tr_k \cdot \gamma_{kt}) \leq b_t \quad t = 1, 2, \ldots, T \]

\[ w_{kt}^{\ell} \leq u_{kt}^{\ell} - u_{kt}^{\ell-1} \quad t = 1, 2, \ldots, T; \quad \ell = 1, 2, \ldots, L; \quad k = 1, 2, \ldots, K \]
MIP based Heuristic

Piecewise Linear Approximation

\[
\begin{align*}
\sum_{\ell=1}^{L} w_{\ell kt} - \sum_{\ell=1}^{L} w_{k,t-1} & = q_{kt} & t = 1, 2, \ldots, T \\
q_{kt} & \leq u_{kt} \cdot \gamma_{kt} & t = 1, 2, \ldots, T; \\
& & k = 1, 2, \ldots, K
\end{align*}
\]
MIP based Heuristic

Piecewise Linear Approximation

\[
w_{kt}^\ell \leq M \cdot \lambda_{kt}^\ell
\]
\[
\ell = 1, 2, \ldots, L; \\
k = 1, 2, \ldots, K
\]

\[
w_{kt}^\ell \geq (u_{kt}^\ell - u_{kt}^{\ell-1}) \cdot \lambda_{kt}^{\ell+1}
\]
\[
\ell = 1, 2, \ldots, L - 1; \\
k = 1, 2, \ldots, K
\]
MIP based Heuristic

Piecewise Linear Approximation: Service level

\[
\frac{\sum_{i=\tau}^{t} (\Delta_{B_{ki}}^0 + \sum_{\ell=1}^{L} \Delta_{B_{ki}}^\ell \cdot w_{ki}^\ell)}{\sum_{i=\tau}^{t} E\{D_{ki}\}} \leq 1 - \beta_{ck} + (1 - \gamma_{k\tau})
\]

proportion of backorders (< 1)

\[k = 1, 2, \ldots, K; \tau = 1, 2, \ldots, T; t = \tau, \tau + 1, \ldots, T; \sum_{i=\tau}^{t} E\{D_{ki}\} > 0\]

\[\sum_{i=1}^{t} \gamma_{ki} \geq 1 \quad k = 1, 2, \ldots, K; t = 1, 2, \ldots, T; \sum_{i=1}^{t} E\{D_{ki}\} > 0\]
MIP based Heuristic

\[ \gamma_{kt} \in \{0, 1\} \quad t = 1, 2, \ldots, T; k = 1, 2, \ldots, K \]
\[ w_{kt}^{\ell} \geq 0 \quad t = 1, 2, \ldots, T; \ell = 1, 2, \ldots, L; k = 1, 2, \ldots, K \]
\[ \lambda_{kt}^{\ell} \in \{0, 1\} \quad t = 1, 2, \ldots, T; \ell = 1, 2, \ldots, L; k = 1, 2, \ldots, K \]
## Numerical Experiment

### Data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of products</td>
<td>10, 40</td>
</tr>
<tr>
<td>Number of periods</td>
<td>10, 20</td>
</tr>
<tr>
<td>Average utilization</td>
<td>40%, 60%, 80%</td>
</tr>
<tr>
<td>Demand (mean)</td>
<td>$E {D_{kt}} \sim DU(20, 40, 60, 80, 100)$</td>
</tr>
<tr>
<td>Demand (coefficient of variation)</td>
<td>$CV {D_{kt}} \sim DU(0.1, 0.3)$</td>
</tr>
<tr>
<td>Random yield</td>
<td>$BI$ yield with $p_k \sim DU(0.7, 0.8, 0.9)$</td>
</tr>
<tr>
<td>Time between orders</td>
<td>$TBO_k \sim DU(2, 3, 4)$</td>
</tr>
<tr>
<td>Setup costs</td>
<td>$s_k = 1000$</td>
</tr>
<tr>
<td>Holding costs</td>
<td>$h_k = 2s_k/(TBO_k^2 \cdot \sum_{t=1}^{T} E {D_{kt}} / T)$</td>
</tr>
<tr>
<td>Setup time</td>
<td>$ts_k \sim DU(5, 10, 20)$</td>
</tr>
<tr>
<td>Processing time</td>
<td>$tb_k = 1$</td>
</tr>
<tr>
<td>Target fill rate</td>
<td>$\beta_{kc} = 0.9$</td>
</tr>
</tbody>
</table>
Numerical Experiment

Solution Quality

<table>
<thead>
<tr>
<th>[K; T]</th>
<th>Total</th>
<th>$\text{ABC}_\beta^Y$</th>
<th>CG</th>
<th>CG2</th>
<th>F&amp;O</th>
</tr>
</thead>
<tbody>
<tr>
<td>[10; 10]</td>
<td>254</td>
<td>0.0%</td>
<td>-12.5%</td>
<td>-13.0%</td>
<td>-11.3%</td>
</tr>
<tr>
<td>[10; 20]</td>
<td>230</td>
<td>0.0%</td>
<td>-15.2%</td>
<td>-15.4%</td>
<td>-14.7%</td>
</tr>
<tr>
<td>[40; 10]</td>
<td>282</td>
<td>0.0%</td>
<td>-13.2%</td>
<td>-13.3%</td>
<td>-11.4%</td>
</tr>
<tr>
<td>[40; 20]</td>
<td>270</td>
<td>0.0%</td>
<td>-16.1%</td>
<td>-16.2%</td>
<td>-14.6%</td>
</tr>
<tr>
<td>Total</td>
<td>1036</td>
<td>0.0%</td>
<td>-15.0%</td>
<td>-15.1%</td>
<td>-13.6%</td>
</tr>
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</table>

Objective value compared to $\text{ABC}_\beta^Y$ solution
CG2 is best, but F&O is more flexible ($CLSP_L$)
### Numerical Experiment

**Solution Time (sec)**

<table>
<thead>
<tr>
<th>[K; T]</th>
<th>Total</th>
<th>$\text{ABC}_\beta^Y$</th>
<th>CG</th>
<th>CG2</th>
<th>F&amp;O</th>
</tr>
</thead>
<tbody>
<tr>
<td>[10; 10]</td>
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<td>0.01</td>
<td>0.47</td>
<td>0.38</td>
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<td>[10; 20]</td>
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<tr>
<td>[40; 10]</td>
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<td>2.33</td>
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<tr>
<td>[40; 20]</td>
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<td>1007.80</td>
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<tr>
<td><strong>Total</strong></td>
<td><strong>1200</strong></td>
<td><strong>0.04</strong></td>
<td><strong>7.34</strong></td>
<td><strong>8.2</strong></td>
<td><strong>347.93</strong></td>
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</table>
### Numerical Experiment

Number of Feasible Solutions

<table>
<thead>
<tr>
<th>$[K; T]$</th>
<th>Total</th>
<th>$\text{ABC}^Y_\beta$</th>
<th>CG</th>
<th>CG2</th>
<th>F&amp;O</th>
<th>CPLEX</th>
</tr>
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<tbody>
<tr>
<td>[10; 10]</td>
<td>300</td>
<td>298</td>
<td>254</td>
<td>298</td>
<td>299</td>
<td>265</td>
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<tr>
<td>[10; 20]</td>
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<td>230</td>
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<td>300</td>
<td>278</td>
</tr>
<tr>
<td>[40; 10]</td>
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<td>300</td>
<td>282</td>
<td>300</td>
<td>300</td>
<td>106</td>
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<td>[40; 20]</td>
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<td>300</td>
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<td>0</td>
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<tr>
<td>Total</td>
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<td>1198</td>
<td>1197</td>
<td>649</td>
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# Numerical Experiment

## CPLEX versus F&O

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<th>CPLEX</th>
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<td>265</td>
<td>-1.3%</td>
</tr>
<tr>
<td>[10; 20]</td>
<td>278</td>
<td>1.3%</td>
</tr>
<tr>
<td>[40; 10]</td>
<td>106</td>
<td>4.5%</td>
</tr>
<tr>
<td>[40; 20]</td>
<td>0</td>
<td>–</td>
</tr>
</tbody>
</table>
In MRP and OM textbooks random yield is almost neglected.
Column generation approach performs best, but
Linearization approach can be extended (e.g. include setup carry-overs: CLSP-L)
By-product of our research: a new variant of ABC heuristic (setup times)
Multi-level extensions