Lead Time Distribution of Three-Machine Two-Buffer Lines with Unreliable Machines and Finite Buffers

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Introduction

- *[Production] Lead Time:* the amount of time from when a part enters the first machine in a line to when it leaves the last machine. *(Also called cycle time.)*
  - Random because of machine failures and other stochastic phenomena.
- *Research Goal:* to calculate $\textbf{prob}(T = \tau)$ where T is the time a part spends in a 3M2B Buzacott-type line and $\tau = 0, 1, 2, \ldots$; to calculate the minimum τ such that $\textbf{prob}(T \leq \tau) \geq 1 - \epsilon$ for specified ϵ.
- *Importance:*
  - Customers demand *short* and *reliable* lead times.
  - The value of some products decreases with lead time.

Introduction

Prior literature

- Several papers obtain lead time distributions (or only moments of distributions) in the classical queueing theory framework: random service times, infinite buffers.
- Others study lead time in models more appropriate to manufacturing:
  - Tan (2003) describes a methodology for calculating performance measures (including lead time) for a wide variety of Buzacott-type lines with finite buffers and different control policies.
  - Shi and Gershwin (2012) obtain an exact distribution of the lead time in a 2M1B Buzacott-type line with a finite buffer, and an approximate distribution of the time in a single buffer of a long line.
  - Shi (2012) studies a buffer allocation problem in which the time a part spends in a single given buffer must satisfy $\textbf{prob}(T \leq \tau) \geq 1 - \epsilon$.
  - Colledani, Angius, and Horvath (2014) extend Shi and Gershwin (2012) to 2M1B lines with general Markovian machines.
  - Biller, Meerkov, and Yan (2013) choose the parameters of the first machine of a line with Bernoulli machines and infinite buffers to maximize the production rate subject to an average lead time constraint. Meerkov and Yan (2014) extend this to lines with exponentially distributed up- and down-times.

Model

Two Dynamic Systems

To determine the probability distribution of the lead time *in steady state,* we study two dynamic systems:

- The Material Flow System Model
  - This is a standard Buzacott-type model of a 3M2B line with two-state machines and finite buffers. The state space is the set of all $(\nu_1, \nu_2, \alpha_1, \alpha_2, \alpha_3)$ where ν_i is the number of parts in Buffer i $(\nu_i = 0, 1, ..., N_i)$ and α_i is the repair state of Machine i. $(\alpha_i = 1$ or 0, *i.e.* operational or under repair.)
- The Reference Part Movement System Model
  - This represents the movement of a *reference part* that enters the system when the system is in state $(\nu_1, \nu_2, \alpha_1, \alpha_2, \alpha_3)$.

Details can be found in the conference paper or in a longer paper in preparation (Shi and Gershwin, 2015).

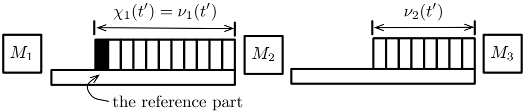
Model

The Reference Part Movement System (RPMS) Model

- *Assumption:* First in, first out buffers.
- *Definition:* The *position* of a part in a buffer is one more than the number of parts in the buffer that will leave the buffer before it.
- *Notation:* $\chi_i(t)$ is the position of the reference part if it is in Buffer i at time t. ($\chi_i(t)$ is not defined if the reference part is in the other buffer.)
- *State of the RPMS:*
  - It does not include anything *upstream* of the reference part.
  - $(\chi_1, \nu_2, \alpha_2, \alpha_3)$ when the reference part is in B_1 ($\chi_1 \leq \nu_1$);
  - (χ_2, α_3) when the reference part is in B_2 ($\chi_2 \leq \nu_2$).

Model

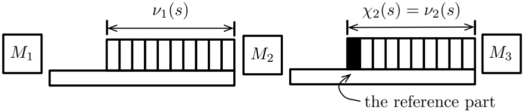
Evolution of the state: Phase 1



- The reference part can enter the system only when M_1 is up and not blocked.
- If the reference part arrives at time t', it goes into B_1 at position $\chi_1(t')$. At that time, $\chi_1(t') = \nu_1(t')$.
- $\chi_1(t)$ decreases as M_2 moves earlier parts from B_1 to B_2 until $\chi_1(t) = 1$, i.e., until the reference part is the first part in B_1.

Model

Evolution of the state: Phase 2



- The reference part moves into B_2 at time s. $\chi_2(s) = \nu_2(s)$.
- $\chi_2(t)$ decreases as M_3 moves earlier parts in B_2 out of the system, i.e., until the reference part is the first part in B_2.
- As soon as M_3 is up, at time u, the reference part leaves the system, and one sample of the lead time ($T = u - t'$) is generated.

Formulation

Definitions

- $A(t)$ is the event that the reference part enters Buffer 1 at time t.
- $\textbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$ is the conditional probability that the reference part has lead time $T = \tau$ given that $\chi_1(t) = x_1$, $\nu_2(t) = n_2$, $\alpha_2(t) = a_2$, $\alpha_3(t) = a_3$, and given that it arrived at time t.
- $\textbf{prob}(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3 | A(t))$ is the conditional probability that $\chi_1(t) = x_1$, $\nu_2(t) = n_2$, $\alpha_2(t) = a_2$, $\alpha_3(t) = a_3$, given that the reference part arrived at time t.

Formulation

Law of Total Probability

$$\textbf{prob}(T = \tau) = \textbf{prob}(T = \tau | A(t)) = \sum_{x_1=1}^{N_1} \sum_{n_2=0}^{N_2} \sum_{a_2=0}^{1} \sum_{a_3=0}^{1} \Big[$$

$$\textbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t)) \times$$
$$\textbf{prob}(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3 | A(t)) \Big]$$

Formulation

Notation Simplification

To make the slides a little easier to read, we define

$$\mathbf{p}(\tau | x_1, n_2, a_2, a_3, A(t)) =$$

$$\mathbf{prob}(T = \tau | \chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$$

$$\mathbf{p}(x_1, n_2, a_2, a_3 | A(t)) =$$

$$\mathbf{prob}(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3 | A(t))$$

Formulation

Law of Total Probability

The Law of Total probability then becomes:

$$\mathbf{prob}(T = \tau) = \mathbf{prob}(T = \tau | A(t)) =$$

$$\sum_{x_1=1}^{N_1} \sum_{n_2=0}^{N_2} \sum_{a_2=0}^{1} \sum_{a_3=0}^{1} \mathbf{p}(\tau | x_1, n_2, a_2, a_3, A(t)) \; \mathbf{p}(x_1, n_2, a_2, a_3 | A(t))$$

Solution

$\mathbf{p}(\tau|x_1, n_2, a_2, a_3, A(t)))$

- We determine this quantity by considering the change in state and lead time in one time step. That is, we relate **prob**$(T = \tau|\text{state at time } t)$ to **prob**$(T = \tau - 1|\text{state at time } t+1)$ for all states that can be reached in one time step.
- We only have to evaluate this probability for states in which the reference part is in the first buffer. This is because the lead time starts when it arrives, and it arrives at the first buffer.
- There are many cases. We consider only one generic case here.
- *Further simplified notation:* If $2 \leq x_1 \leq N_1$,

$$\Pi_t^{a_2 a_3}(\tau, x_1, n_2) = \mathbf{p}(\tau|x_1, n_2, a_2, a_3, A(t)))$$

- New notation for $x_1 = 1$ is defined in the paper.

Solution

p$(\tau | x_1, n_2, a_2, a_3, A(t)))$

Then $$\Pi^{11}(\tau, x_1, n_2) =$$

$$\begin{array}{lcl} p_2 p_3 \Pi^{00}(\tau - 1, x_1, n_2) & + & p_2(1 - p_3)\Pi^{01}(\tau - 1, x_1, n_2 - 1) \\ +(1 - p_2)p_3 \Pi^{10}(\tau - 1, x_1 - 1, n_2 + 1) & + & (1 - p_2)(1 - p_3)\Pi^{11}(\tau - 1, x_1 - 1, n_2) \end{array}$$

- $p_2 p_3 \Pi^{00}(\tau - 1, x_1, n_2)$: M_2 and M_3 fail so reference part position and Buffer 2 level do not change.
- $p_2(1 - p_3)\Pi^{01}(\tau - 1, x_1, n_2 - 1)$: M_2 fails, M_3 stays up so reference part position does not change and Buffer 2 loses a part.
- $(1 - p_2)p_3\Pi^{10}(\tau - 1, x_1 - 1, n_2 + 1)$: M_2 stays up, M_3 fails so reference part position decreases and Buffer 2 level increases.
- $(1 - p_2)(1 - p_3)\Pi^{11}(\tau - 1, x_1 - 1, n_2)$: Both machines stay up so reference part position decreases and Buffer 2 level stays constant.

Solution

p$(\tau | x_1, n_2, a_2, a_3, A(t))$)

- Other equations, including initial conditions, are derived similarly.
- When $x_1 = 1$ and M_2 is up and not blocked, the reference part with leave B_1 in the next time step and enter B_2. The probability distribution of the time it spends in B_2 is given by the 2M1B lead time distribution of (Shi and Gershwin, 2012).
- All these conditional probabilities are now determined.
- They are the first set of factors in the Law of Total Probability.

Solution

$\mathbf{p}(x_1, n_2, a_2, a_3|A(t))$

These quantities can be expressed in terms of the steady-state probability distribution of the 3M2B line $\mathbf{p}(n_1, n_2, a_1, a_2, a_3)$.

1. When the reference part enters the line, $\chi_1(t) = x_1 = \nu_1(t) = n_1$. Therefore

$$\mathbf{p}(x_1, n_2, a_2, a_3|A(t)) =$$

$$\mathbf{prob}(\chi_1(t) = x_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t)) =$$

$$\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t))$$

2. When the reference part enters the line, the first machine must be up. Therefore

$$\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t)) =$$

$$\mathbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t))$$

Solution

$\mathbf{p}(x_1, n_2, a_2, a_3|A(t))$

3. From the definition of conditional probability,

$$\textbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3|A(t))$$

$$= \frac{\textbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))}{\textbf{prob}(A(t))}$$

4. $A(t)$, the event that a part enters the line, is the event that the first machine is up and not blocked, which is easy to calculate from the steady-state probability distribution of the 3M2B line.

   In fact, $\textbf{prob}(A(t))$ is the production rate of the line.

Solution

$\mathbf{p}(x_1, n_2, a_2, a_3|A(t))$

5. Shi and Gershwin (2015) provide expressions for

   $$\textbf{prob}(\nu_1(t) = n_1, \nu_2(t) = n_2, \alpha_1(t) = 1, \alpha_2(t) = a_2, \alpha_3(t) = a_3, A(t))$$

   in terms of $\mathbf{p}(n_1, n_2, a_1, a_2, a_3)$.

We now have the second set of factors in the Law of Total Probability, and therefore enough to evaluate $\textbf{prob}(T = \tau)$.

Numerical Experiments

1. Check with Little's Law

There are two ways of calculating the mean lead time:

$$\text{From Little's law: } \mathbf{E}[T] = \frac{\bar{n}_1 + \bar{n}_2}{P}$$

$$\text{From the lead time distribution: } \mathbf{E}[T] = \sum_{\tau=0}^{\infty} \tau \, \mathbf{prob}(T = \tau)$$

We use Tan's (2003) Matlab code to calculate the steady state distribution of a 3M2B line for our numerical examples.

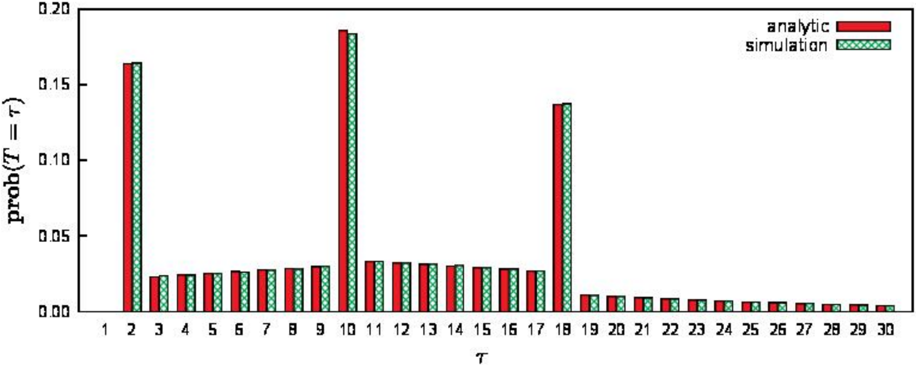
Numerical Experiments

1. Check with Little's Law

Case	1	2	3	4	5
r_1, p_1	.1, .01	.8, .096	.07, .01	.2, .02	.12, .009
r_2, p_2	.1, .01	.1, .01	.12, .008	.2, .02	.15, .009
r_3, p_3	.1, .01	.1, .01	.12, .008	.4, .048	.07, .01
N_1	10	30	16	18	19
N_2	10	22	23	35	17
P	.819137	.861210	.847203	.874546	.848478
$\bar{n}_1$	5.983370	14.496551	6.606718	9.860994	12.468434
$\bar{n}_2$	4.016630	9.393820	6.665852	16.534348	9.779282
$(\bar{n}_1 + \bar{n}_2)/P$	12.207969	27.740476	15.666334	30.181748	26.220720
$\mathbf{E}[T]$	12.207969	27.740476	15.666334	30.181748	26.220720

Numerical Experiments

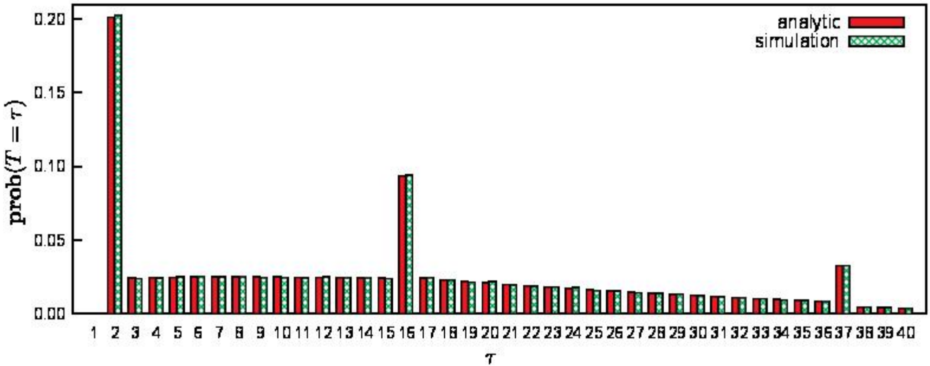
2. Comparison with Simulation



$p_i = .01$ and $r_i = .1$ for $i = 1, 2, 3,\ N_1 = N_2 = 10$

Numerical Experiments

2. Comparison with Simulation

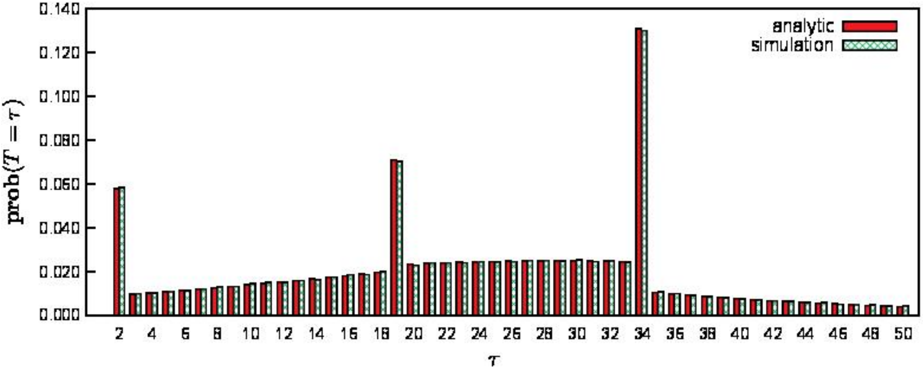


i	p_i	r_i	e_i	N_i
1	.01	.07	.875*	16
2	.008	.12	.938	23
3	.008	.12	.938	

** Bottleneck*

Numerical Experiments

2. Comparison with Simulation



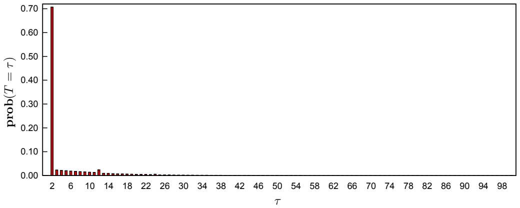
i	p_i	r_i	e_i	N_i
1	.009	.12	.930	19
2	.009	.15	.943	17
3	.01	.07	.875*	

* *Bottleneck*

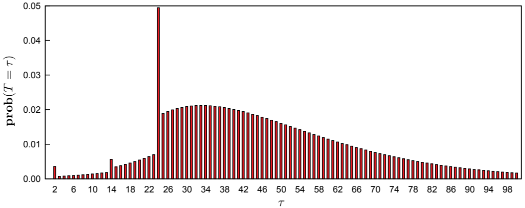
Numerical Experiments

3. A Line and Its Reverse

	r_1	p_1	r_2	p_2	r_3	p_3	N_1	N_2	P	$\bar{n}_1$	$\bar{n}_2$
original line	.1	.1*	.1	.01	.1	.01	12	14	.493214	1.284711	1.293355
reversed line	.1	.01	.1	.01	.1	.1*	14	12	.493214	12.706645	10.715289



(a) Lead time distribution of the original line

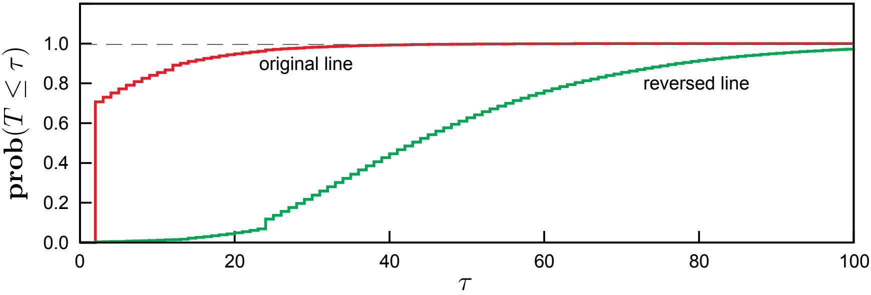


(b) Lead time distribution of the reversed line

* *Bottleneck*

Numerical Experiments

3. A Line and Its Reverse



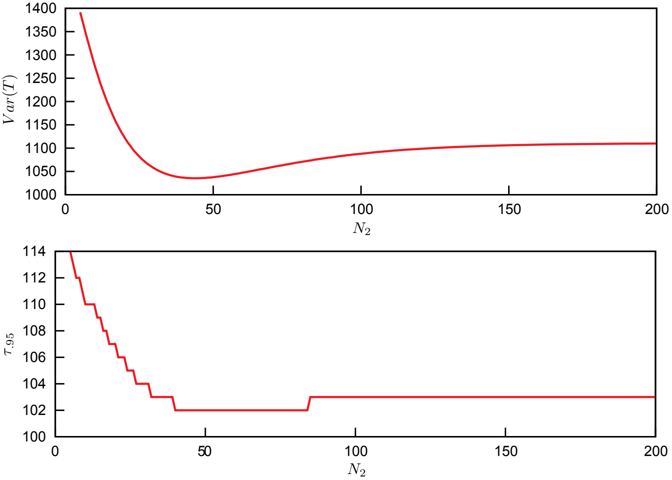
Numerical Experiments

$\tau_{.95}$

In the following slides, $\tau_{.95}$ is the minimum value of τ such that **prob**$(T \leq \tau) \geq .95$.

Numerical Experiments

4. Variance and $\tau_{.95}$ vs. N_2



$r_1 = .07,\ p_1 = .01,\ r_2 = .12,\ p_2 = .008,\ r_3 = .12,\ p_3 = .008,\ N_1 = 100$

Numerical Experiments

5. Variance and $\tau_{.95}$ such that $\textbf{prob}(T \leq \tau) \geq .95$ vs. MTBF_i

- In the following graphs, we vary r_i and p_i together so that

  $$e_i = \frac{r_i}{r_i + p_i}$$

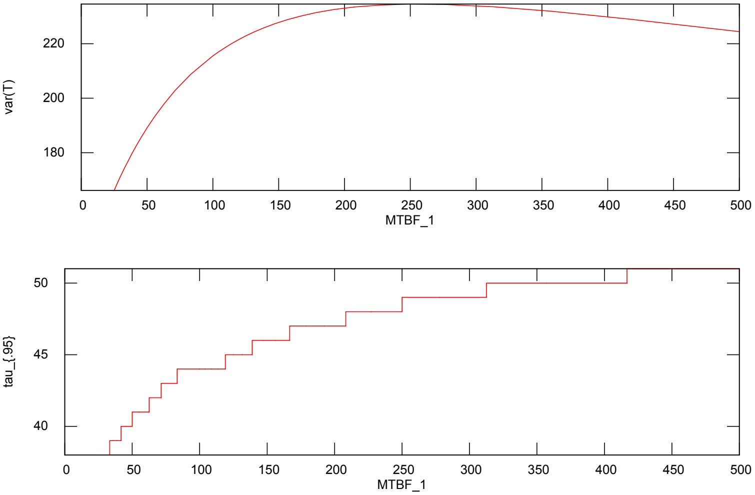
  is constant.

- The x axis is

  $$\text{MTBF}_i = \text{MTTR}_i + \text{MTTF}_i = \frac{1}{r_i} + \frac{1}{p_i}$$

Numerical Experiments

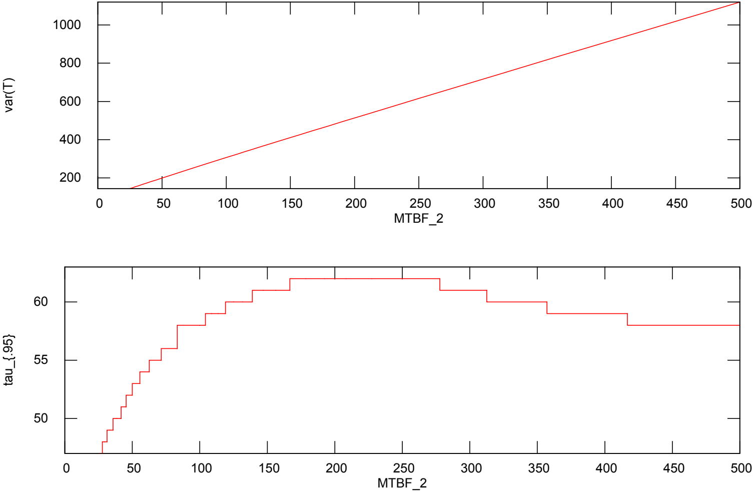
5. Variance and $\tau_{.95}$ vs. $MTTF_1$



$e_1 = .8,\ r_2 = .1,\ p_2 = .01,\ r_3 = .1,\ p_3 = .01,\ N_1 = N_2 = 20$

Numerical Experiments

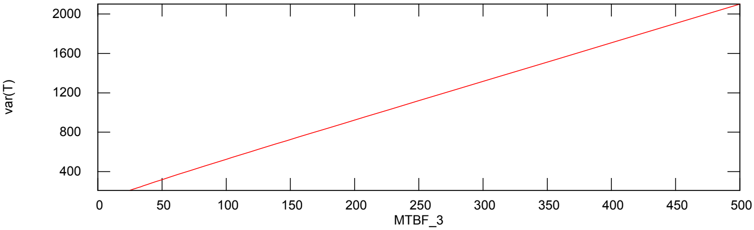
5. Variance and $\tau_{.95}$ MTTF$_2$

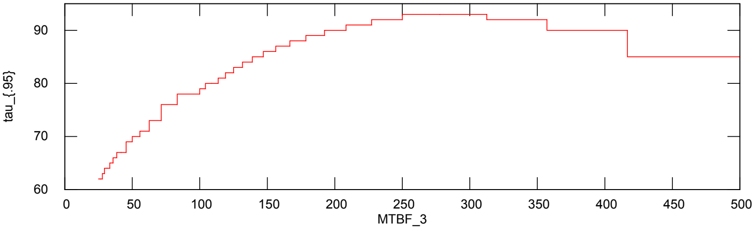


$e_2 = .8, \ r_1 = .1, \ p_1 = .01, \ r_3 = .1, \ p_3 = .01, \ N_1 = N_2 = 20$

Numerical Experiments

4. Variance and $\tau_{.95}$ vs. MTTF_3





$$e_3 = .8,\ r_1 = .1,\ p_1 = .01,\ r_2 = .1,\ p_2 = .01,\ N_1 = N_2 = 20$$

Future Research

- Extensions to longer lines, lines with machines whose repair/failure behavior is described by general Markov chains, and more general system topologies. (See next talk[1].)
- Extensions to LIFO and random buffer disciplines.
- Extensions to systems with continuous time and discrete or continuous material.
- Understand the qualitative behavior of $Var(T)$ and $\tau_{.95}$, and determine their relationships with the classification of 3M2B lines into five classes by Shi and Gershwin (2013).
- See if this analysis can approximate the lead time distribution in a 3M2B segment of a longer line.
- Optimize buffer size and machine choice under a constraint on $\tau_{.95}$.
- Develop an approximation of the lead time for large systems — systems for which it is not feasible to calculate or store the steady-state probability distribution.

[1] *Lead Time Dependent Product Deterioration in Manufacturing Systems with Serial, Assembly and Closed- Loop Layout* presented by Marcello Colledani in this conference; paper by Colledani, Angius, Horvath, and Gershwin.

Thank you. Questions?

