Flow Modulating Effects of Inspection Stations in Manufacturing Lines

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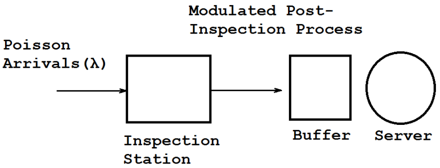
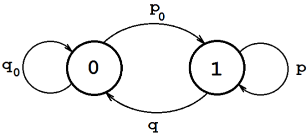
June 2, 2015



- Gershwin (2006) How do quantity and quality really interact?
- Kim and Gershwin (2006) Integrated quality and quantity modeling of a production line.
- Colledani and Tolio (2011) Integrated analysis of quality and production logistics performance in manufacturing lines.
- Tapiero and Hsu (1987) Quality control of the M/M/1 Queue.
- Tsiotras and Tapiero (1992) WIP and CSP-1 quality control in a tandem queueing production system.

- During the full inspection phase inspect every part
- After finding k consecutive good parts switch to sampling every one out of r parts. (Bernoulli sampling with sampling probability $\alpha = 1/r$.

Markovian Quality process (0 = Defective Part, 1 = Good Part)



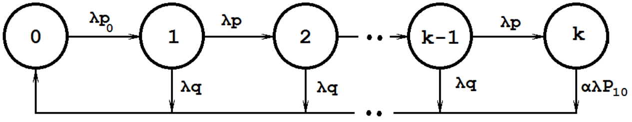
$$Q = \begin{bmatrix} q_0 & p_0 \\ q & p \end{bmatrix}$$

Bernoulli sampling with probability $\alpha = 1/r$. The transition probability matrix for the quality process after sampling:

$$P := \sum_{n=1}^{\infty} \alpha(1-\alpha)^{n-1} Q^n = \alpha Q(I - (1-\alpha)Q)^{-1}.$$

$$P := \frac{1}{1-(1-\alpha)(p-p_0)} \begin{bmatrix} q_0 - (1-\alpha)(p-p_0) & p_0 \\ q & p-(1-\alpha)(p-p_0) \end{bmatrix}.$$

Parts arrive according to a Poisson process (λ). $\{X_t; t \geq 0\}$ is the state of the inspector. (CTMC)

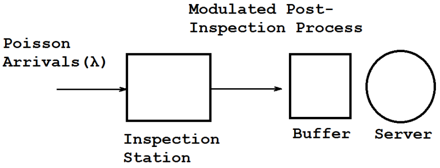


The stationary distribution of the chain is

$$\begin{aligned} \pi_i &= \pi_0 p_0 p^{i-1}, \qquad i = 1, 2, \ldots, k-1 \\ \pi_k &= \pi_0 p_0 \frac{1}{\alpha P_{10}} p^{k-1}, \\ \pi_0 &= \frac{1}{1 + \frac{p_0}{q} + p_0 \left(\frac{1}{\alpha P_{10}} - \frac{1}{q} \right) p^{k-1}} = \frac{\alpha q}{(q + p_0)[\alpha + (1-\alpha) p_0 p^{k-1}]}. \end{aligned} \tag{1}$$

Intensity of the post-inspection process (according to the state of the inspector

$$\begin{aligned} \nu_0 &= \lambda p_0 \\ \nu_i &= \lambda p, \qquad i = 1, \ldots, k-1 \\ \nu_k &= \lambda(1 - \alpha P_{10}) \end{aligned}$$



$\{N_t; t \geq 0\}$ the Inspection-Modulated Arrival Process.

$$r := \lim_{t\to\infty} \frac{1}{t}\mathbb{E}N_t \tag{2}$$

$$v^2 := \lim_{t\to\infty} \frac{1}{t}\mathrm{Var}(N_t) \tag{3}$$

Let $\{M_t; t \geq 0\}$ be an approximating renewal process with interarrival times $\{\tau_n\}$. If $\mu := \mathbb{E}[\tau]$, $\sigma^2 := \mathrm{Var}(\tau)$,

$$\mu^{-1} = r \tag{4}$$

$$\frac{\sigma^2}{\mu^3} = v^2 \tag{5}$$

Let N_t be the number of arrivals in the interval $(0, t]$. The rate is

$$r = \sum_{i=0}^{k} \nu_i \pi_i = \lambda p_0 \pi_0 + \lambda p(\pi_1 + \cdots + \pi_{k-1}) + \lambda(1 - \alpha P_{10}) = \lambda(1 - \pi_0)$$

From

$$N_t^2 = 2 \int_0^t N_{s-} dN_s + \int_0^t dN_s$$

we have

$$\mathrm{Var}(N_t) = 2\mathbb{E}\left[\int_0^t \int_0^s \lambda_u \lambda_s du ds\right] + \mathbb{E}\int_0^t \lambda_s ds - \left(\mathbb{E}\int_0^t \lambda_s ds\right)^2 .$$

Thus, since for $u < s$ $\mathbb{E}[\lambda_u \lambda_s] = \sum_{i,j} \nu_i \nu_j \pi_i P_{ij}(s-u)$, where $P_{ij}(s) = \mathbb{P}(X_s = j | X_0 = i)$

$$\begin{aligned}
\frac{1}{t}\mathrm{Var}(N_t) &= \frac{2}{t}\mathbb{E}\left[\int_0^t \int_0^s \sum_{i,j} \nu_i \nu_j \pi_i P_{ij}(s-u) du ds\right] + \sum_i \pi_i \nu_i - t \sum_{i,j} \nu_i \nu_j \pi_i \pi_j \\
&= \frac{2}{t}\mathbb{E}\left[\int_0^t \int_0^s \sum_{i,j} \nu_i \nu_j \pi_i \left(P_{ij}(s-u) - \pi_j\right) du ds\right] + \sum_i \pi_i \nu_i \\
&= \sum_{i,j} \nu_i \nu_j \pi_i \frac{2}{t} \int_0^t \int_0^s \left(P_{ij}(s-u) - \pi_j\right) du ds + \sum_i \pi_i \nu_i
\end{aligned}$$

$$v^2 \quad := \quad \lim_{t\to\infty} \frac{1}{t}\mathrm{Var}(N_t) \; = \; 2\sum_{i,j} \nu_i \nu_j \pi_i \int_0^\infty (P_{ij}(t) - \pi_j)\, dt + \sum_i \pi_i \nu_i \quad (6)$$

The fundamental matrix (or recurrent potential matrix) of an irreducible, positive recurrent Markov Chain

$$Z_{ij} := \int_0^\infty (P_{ij}(t) - \pi_j) dt$$

If $T_j := \inf\{t > 0 : X_t = j\}$ then

$$Z_{jj} \quad = \quad \pi_j \mathbb{E}_\pi[T_j], \qquad (7)$$

$$Z_{ij} \quad = \quad Z_{jj} - \pi_i \mathbb{E}_i[T_j] \qquad (8)$$

Also

$$\sum_j Z_{ij} = 0 \quad \text{and} \quad \sum_i \pi_i Z_{ij} = 0$$

Since $\sum_j Z_{ij} = 0$

$$\begin{aligned}\sum_{i=0}^{k} \nu_i \pi_i \sum_{j=0}^{k} Z_{ij} \nu_j &= \sum_{i=0}^{k} \nu_i \pi_i \left(\sum_{j=0}^{k} Z_{ij}(\lambda p) + Z_{i0}(\lambda p_0 - \lambda p) + Z_{ik}(\lambda(1 - \alpha P_{10}) - \lambda p) \right) \\ &= \lambda(p_0 - p) \sum_{i=0}^{k} \nu_i \pi_i Z_{i0} \; + \; \lambda(q - \alpha P_{10}) \sum_{i=0}^{k} \nu_i \pi_i Z_{ik}. \end{aligned} \tag{9}$$

Also, since $\sum_i \pi_i Z_{ij} = 0$,

$$\begin{aligned}\sum_{i=0}^{k} \nu_i \pi_i Z_{i0} &= \lambda p \sum_{i=0}^{k} \pi_i Z_{i0} + \lambda(p_0 - p)\pi_0 Z_{00} + \lambda(q - \alpha P_{10})\pi_k Z_{k0} \\ &= \lambda(p_0 - p)\pi_0 Z_{00} + \lambda(q - \alpha P_{10})\pi_k Z_{k0}\end{aligned}$$

and $\sum_{i=0}^{k} \nu_i \pi_i Z_{ik} = \lambda(p_0 - p)\pi_0 Z_{0k} + \lambda(q - \alpha P_{10})\pi_k Z_{kk}$ Thus (9) becomes

$$\begin{aligned}\sum_{i=0}^{k} \nu_i \pi_i \sum_{j=0}^{k} Z_{ij} \nu_j &= \lambda^2 (p_0 - p)^2 \pi_0 Z_{00} + \lambda^2 (p_0 - p)(q - \alpha P_{10}) \left(\pi_0 Z_{0k} + \pi_k Z_{k0} \right) \\ &\quad + \lambda^2 (q - \alpha P_{10})^2 \pi_k Z_{kk}.\end{aligned}$$

If $m_{ij} := \mathbb{E}_i[T_j]$ denote the mean transition times

$$Z_{00} = \pi_0 \mathbb{E}_\pi T_0, \qquad Z_{k0} = Z_{00} - \pi_0 m_{k0},$$
$$Z_{kk} = \pi_k \mathbb{E}_\pi T_k, \qquad Z_{0k} = Z_{kk} - \pi_k m_{0k}.$$

and hence $\pi_0 Z_{0k} + \pi_k Z_{k0} = \pi_0 \pi_k \left(\mathbb{E}_\pi T_0 + \mathbb{E} T_k - m_{k0} - m_{0k} \right)$

$$\begin{aligned}\lambda^{-2} \sum_{i=0}^{k} \nu_i \pi_i \sum_{j=0}^{k} Z_{ij} \nu_j &= \pi_0^2 (p_0 - p)^2 \mathbb{E}_\pi T_0 \\ &+ \pi_0 \pi_k (p_0 - p)(1 - \alpha P_{10}) \left(\mathbb{E}_\pi T_0 + \mathbb{E} T_k - m_{k0} - m_{0k} \right) + (q - \alpha P_{10})^2 \pi_k^2 \mathbb{E}_\pi T_0 \\ &= (q - \pi_0)(p_0 - p) \pi_0 \mathbb{E}_\pi T_0 + (q - \alpha P_{10})(q - \pi_0) \pi_k \mathbb{E}_\pi T_k \\ &- \pi_k \pi_0 (p_0 - p)(q - \alpha P_{10})(m_{k0} + m_{0k}).\end{aligned}$$

$$
\begin{aligned}
m_{0i} &= \frac{1}{\lambda q p_0}\left((q+p_0)p^{-i+1} - p_0\right), \quad i = 1, 2, \ldots, k \\
m_{ij} &= \frac{q+p_0}{\lambda q p_0}\left(p^{-j+1} - p^{-i+1}\right), \quad 0 < i < j \leq k \\
m_{ij} &= \frac{1}{\lambda q}\left(p^{-j} + \frac{1-\alpha}{\alpha}p^{k-i}\right), \quad i > j \\
m_{kj} &= \frac{1}{\lambda q}\left(\frac{1-\alpha}{\alpha} + p^{-j}\right) \\
m_{i0} &= \frac{1}{\lambda q}\left(1 + \frac{1-\alpha}{\alpha}(q+p_0)p^{k-i}\right).
\end{aligned}
$$

The rate of the modulated input process is

$$r = \lambda p_0 \, \frac{\alpha + (1-\alpha)(q+p_0)p^{k-1}}{\alpha(q+p_0) + (1-\alpha)p_0 p^{k-1}}.$$

Its variance constant, $v^2 := \lim_{t\to\infty} \frac{1}{t}\mathrm{Var}(N_t)$ is

$$\begin{aligned} v^2 \quad = \quad & 2\lambda \frac{p_0}{q}\pi_0^2(q-\pi_0)\left[\frac{p_0-p}{q}\right. \\ & + \left(\frac{1}{\alpha P_{10}} - \frac{1}{q}\right)\left[\frac{(q+p_0)^2}{p_0} - kqp^{k-1} - \left(1 - \frac{q(p_0-p)}{\alpha P_{10}}\right)p^{k-1}\right]\Bigg] \\ & -\lambda q \pi_0 (p_0-p)\left(\frac{1}{\alpha P_{10}} - \frac{1}{q}\right) + \lambda(1-\pi_0) \end{aligned}$$

In this case $P_{10} = q$ and hence $\pi_0 = \frac{q\alpha}{\alpha+(1-\alpha)p^k}$, $\pi_k = \frac{\pi_0 p^k}{\alpha q} = \frac{p^k}{\alpha+(1-\alpha)p^k}$. The rate of the process is

$$r = \lambda p \, \frac{\alpha + (1-\alpha)p^{k-1}}{\alpha + (1-\alpha)p^k}.$$

The variance constant, $v^2 := \lim_{t\to\infty} \frac{1}{t}\mathrm{Var}(N_t)$, is given by

$$v^2 \quad = \quad \lambda p \, \frac{\alpha + (1-\alpha)p^{k-1}}{\alpha + (1-\alpha)p^k} + 2\lambda \frac{q\alpha(1-\alpha)^2}{(\alpha + (1-\alpha)p^k)^3} \, p^k \left(1 - (kq+1)p^k\right).$$

In this case, $P_{10} = q$ (again) and therefore

$$r = \lambda \frac{p_0}{p_0 + q}$$

and

$$v^2 = 2\lambda \frac{1}{q^2} \pi_0^2 p_0 (q - \pi_0)(p_0 - p) + \lambda(1 - \pi_0)$$

which gives

$$v^2 = 2\lambda \frac{p_0 q (p - p_0)^2}{(q + p_0)^3} + \lambda \frac{p_0}{q + p_0}.$$

A two moment approximation

Let $\{M_t; t \geq 0\}$ be a renewal process with interevent times $\{\tau_n\}$, $n = 1, 2, \ldots$, such that $\mathbb{E}[\tau_1] = \mu$, $\mathrm{Var}(\tau_1) = \sigma^2$. We set

$$\mu = \frac{1}{\lambda p} \frac{\alpha + (1-\alpha)p^k}{\alpha + (1-\alpha)p^{k-1}}$$

Set

$$v^2 \quad := \quad \lim_{t\to\infty} \frac{1}{t}\mathrm{Var}(M_t) = \frac{\sigma^2}{\mu^3}$$

Thus

$$\sigma^2 \quad = \quad \frac{1}{\lambda^2 p^2} \left(\frac{\alpha(1-\alpha)p^k}{\alpha + (1-\alpha)p^{k-1}} \right)^2 + \frac{2q\alpha(1-\alpha)^2}{\lambda^2 p^3} \frac{p^k\,(1-(kq+1)p^k)}{(\alpha + (1-\alpha)p^{k-1})^3}$$

We will use the above together with the two moment approximation for the waiting time

$$\text{Mean Waiting Time} \approx \frac{\rho^2(1+C_s^2)}{1+\rho^2 C_s^2} \frac{C_a^2 + \rho^2 C_s^2}{2\lambda(1-\rho)}$$

The two moment approximation when $\alpha = 1$ Here

$$\mu = \frac{1}{\lambda} \frac{p_0 + q}{p_0}$$

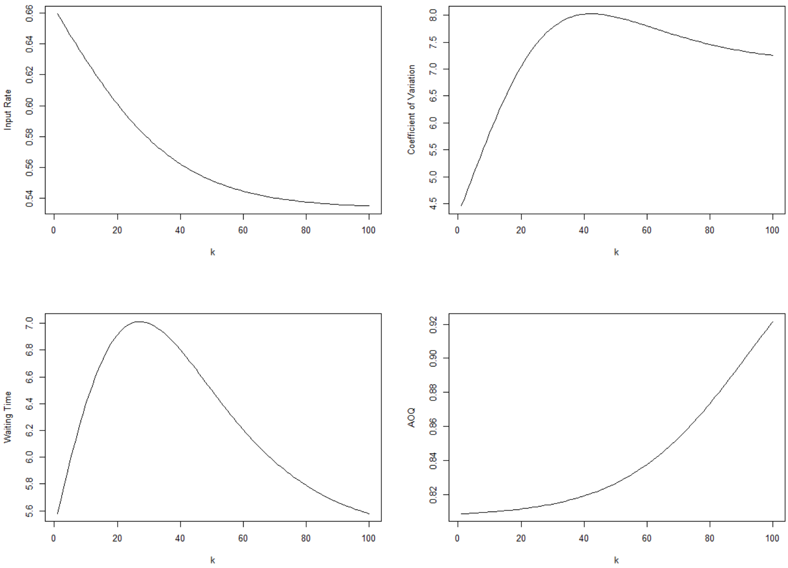
and

$$v^2 = 2\lambda p_0 q \frac{(p - p_0)^2}{(q + p_0)^3} + \lambda \frac{p_0}{p_0 + q},$$

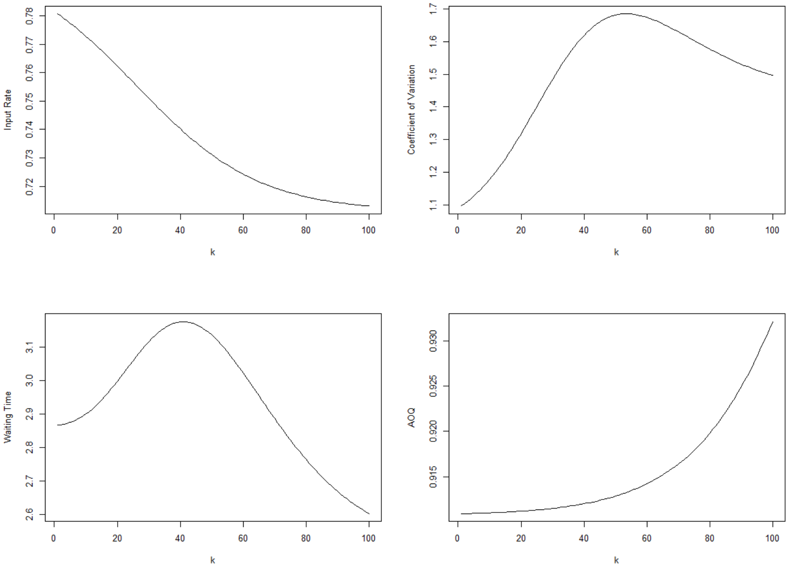
therefore

$$\sigma^2 = \frac{1}{\lambda^2 p_0^2} \left(2q(p - p_0)^2 + (p_0 + q)^2 \right).$$

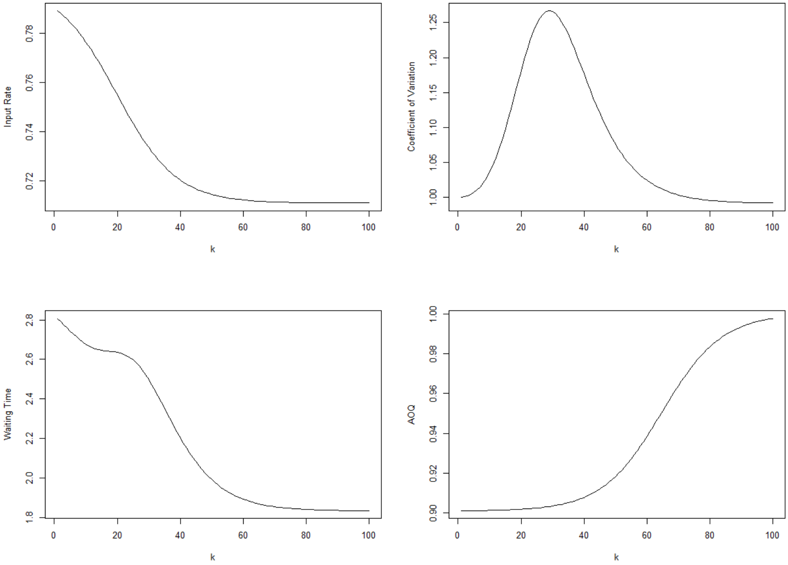
Numerical Results: $p = 0.95,\ a = 0.1,\ p_0 = 0.1,\ \lambda = 0.8,\ C_s^2 = 0.5$



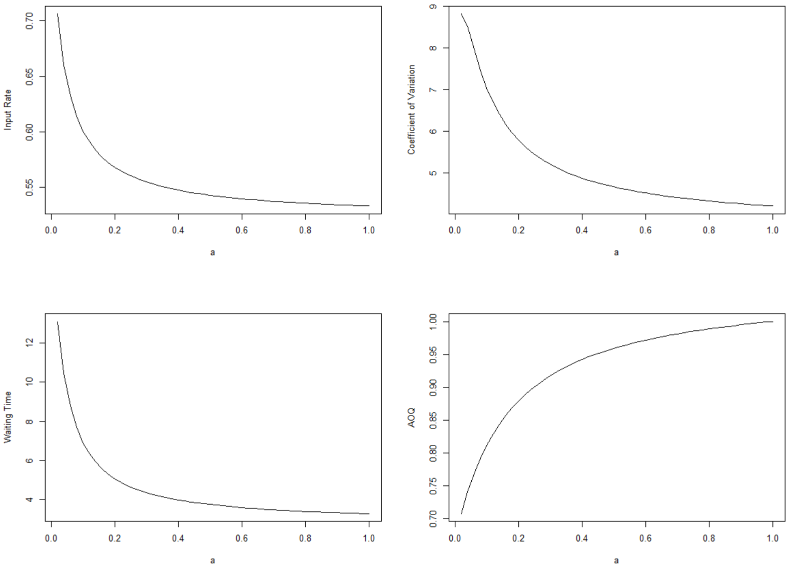
Numerical Results: $p = 0.95,\ a = 0.1,\ p_0 = 0.4,\ \lambda = 0.8,\ C_s^2 = 0.5$



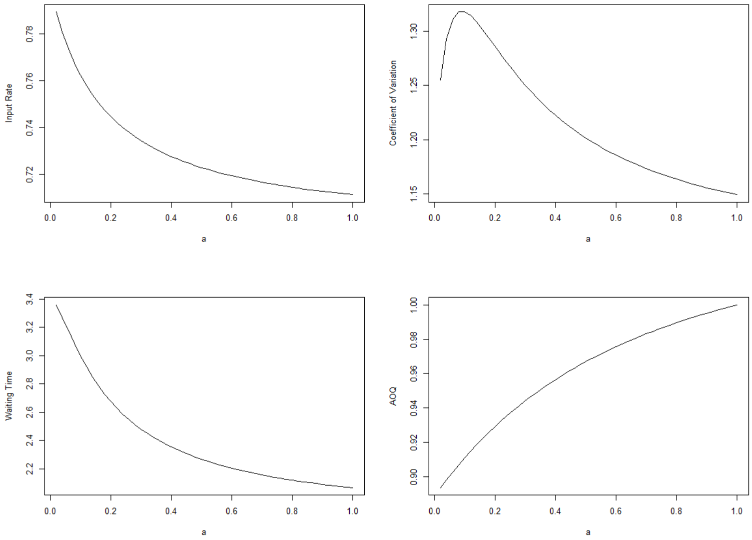
Numerical Results: $p = 0.9,\ a = 0.1,\ p_0 = 0.8,\ \lambda = 0.8,\ C_s^2 = 0.5$



Numerical Results: $p = 0.95,\ k = 20,\ p_0 = 0.1,\ \lambda = 0.8,\ C_s^2 = 0.5$



Numerical Results: $p = 0.95,\ k = 20,\ p_0 = 0.4,\ \lambda = 0.8,\ C_s^2 = 0.5$



A CTMP with state space $S := \{(i,j) : i = 0,1,2,\ldots; j = 0,1,\ldots,k\}$.
The first coordinate i is the *level* of the process while the second coordinate, j, is the *phase*. The generator has the form

$$Q = \begin{bmatrix} B_0 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & & \end{bmatrix}. \tag{10}$$

A_0, A_1, A_2, B_0, are $(k+1)\times(k+1)$ matrices which satisfy the necessary and sufficient conditions in order for Q to be a generator.
All the entries of A_0 and A_2 and all non–diagonal entries of A_1 and B_0 must be non–negative.
If e is a column vector in $\mathbb{R}^{k+1}$ with all entries equal to 1 we should have $(A_0 + B_0)e = 0$ and $(A_0 + A_1 + A_2)e = 0$.
The matrices A_m, $m = 0,1,2$, and B_0 are such that the generator Q is irreducible.

Theorem

Positive Recurrence - Matrix Geometric Form The process with infinitesimal generator Q is positive recurrent if and only if the minimal nonnegative solution R to the matrix equation

$$A_0 + RA_1 + R^2 A_2 = 0 \tag{11}$$

has all its eigenvalues inside the unit disc and the finite system of equations

$$\begin{aligned} p_0(B_0 + RA_2) &= 0 \\ p_0(I - R)^{-1}e &= 1 \end{aligned} \tag{12}$$

has a unique positive solution p_0. Then the stationary probability corresponding to Q is given in partitioned form by $p = [p_0, p_1, p_2, \ldots]$ where p_i, $i = 1, 2, \ldots$, are row vectors with $k+1$ components, $p_i = (p_{i,0}, p_{i,1}, \ldots, p_{i,k})$, corresponding to the $k+1$ different states for the arrival process. They are given by

$$p_i = p_0 R^i, \qquad n = 0, 1, 2, \ldots. \tag{13}$$

A simple criterion that allows one to determine whether Q is positive recurrent. ($\mathrm{sp}(R)$ denotes the spectral radius of the matrix R.)

Theorem

If the matrix $A := A_0 + A_1 + A_2$ is irreducible, then $\mathrm{sp}(R) < 1$ if and only if

$$\pi A_0 e < \pi A_2 e \tag{14}$$

where π is the stationary probability vector of A, i.e. the probability vector that satisfies $\pi A = 0$.

The matrix-geometric form leads to simple expressions for various marginal distributions and moments associated with the stationary distribution.
The marginal probability distribution of the phase given by the vector

$$\sum_{n \geq 0} p_n = p_0 \left(I - R\right)^{-1}$$

while the marginal distribution of the level is given by

$$\pi_n = p_n e = p_0 R^n e, \qquad n = 0, 1, 2, \ldots.$$

The phase correspond to the state of the inspector, i.e. whether the inspection is in sampling or in full inspection mode, and, in the latter case, it indicates the size of the last run of consecutive good parts it has detected.
The level corresponds to the number of parts in the queue (before or after the inspector, according to the model).
Once the matrix R has been determined, one can compute various performance criteria of interest. The expected number of parts in the system is given by

$$L := \sum_{i=1}^{\infty} i p_i e = p_0 \left(\sum_{i=1}^{\infty} i R^i \right) e.$$

(In the above expression the infinite sum of matrices converges by virtue of the fact that $\mathrm{sp}(R) < 1$.) In view of the identity $\left(\sum_{i=1}^{\infty} i R^{i-1}\right)(I - R) = \sum_{i=0}^{\infty} R^i = (I - R)^{-1}$ we have

$$L = p_0 (I - R)^{-2} R\, e.$$

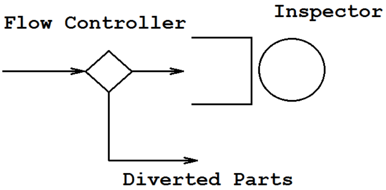
The variance for the queue length is given by

$$V := \sum_{i=1}^{\infty} i^2 p_i e - L^2 = p_0 \left(2(I - R)^{-3} R^2 + (I - R)^{-2} R \right) e - L^2.$$

- When the time to inspect an item is not negligible compared to the other processing times, then it also becomes a consideration in flow–time analysis.
- We consider the inspection station as a single server queue with infinite buffer space. The decision to inspect or not has thus implications not only on the quality of the output but on the flow–time as well.
- Items arrive at this station according to a Poisson process (λ).
- Inspection times are assumed to be i.i.d. exponential with rate μ.
- We analyze two variations of this situation, the *Diverted Stream Model* and the *Accelerated Inspection Model.*

1. The *Diverted Stream Model.*
   Suppose that the quality process is Bernoulli and assume that when the inspector is in the full inspection mode all items join the inspector queue. When the inspector switches to the sampling inspection mode then each arriving item independently joins the queue with probability $\alpha := 1/r$ or moves downstream without being inspected with probability $1 - 1/r$.
2. The *Accelerated Inspection Model.*
   This is introduced in order to facilitate the analysis of a *Markovian quality process*. In this model *all items join the inspector queue, even when the inspector operates in its sampling mode.*
   When the inspector operates in the full inspection phase inspection times are exponential random variables with rate μ.
   When the inspector switches to the sampling phase, the inspection rate becomes $r\mu$ while at the same time a defective item which is inspected is detected as such with probability $1/r$.
   This model is intended as an approximation to the first which describes more accurately the operation of the system.

Diverted stream model



- Items arrive according to a Poisson process (λ). The quality process is Bernoulli (the probability that a part is good is p).
- A part that has joined the inspector queue it is inspected in a FIFO fashion by the inspector (mean inspection time $1/mu$).

This model can be described by a continuous time Markov chain with state space $S := \{(i,j) : i = 0, 1, 2, \ldots, j = 0, 1, \ldots, k\}$ where the first coordinate i, designates the *level i.e.* the number of customers in the system while the second, j, the *phase* in the markovian description of the inspector as described in the previous section.

The generator, Q, is given by (10) where A_i, for $i = 0, 1, 2$, and B_0 are $(k+1) \times (k+1)$ matrices given by

$$B_0 = -A_0, \quad A_0 = \begin{bmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \ddots & & \\ & & & \lambda & \\ & & & & \lambda r^{-1} \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(\lambda+\mu) & & & \\ & -(\lambda+\mu) & & \\ & & \ddots & \\ & & & -(\lambda r^{-1}+\mu) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \mu q & \mu p & & & \\ \mu q & & \mu p & & \\ & & & \ddots & \\ \mu q & & & & \mu p \\ \mu q & & & & \mu p \end{bmatrix}$$

For the stability condition for this model matrix A is given by $A = A_0 + A_1 + A_2$ or:

$$A = \mu \begin{bmatrix} -p & p & & & & \\ q & -1 & p & & & \\ & & \ddots & \ddots & & \\ q & & & & -1 & p \\ q & & & & & -q \end{bmatrix}.$$

The solution of the system $\pi A = 0$ where π together with the normalization condition $\sum_{i=0}^{k} \pi_i = 1$ gives

$$\begin{aligned} \pi_i &= p^i q, \qquad i = 0, 1, \ldots, k-1, \\ \pi_k &= p^k. \end{aligned}$$

Condition (14) gives the stability condition for this system.

$$\lambda / \mu < \frac{1}{1 - p^k(1 - r^{-1})}.$$

- Markovian quality process (necessary in order to capture the effect of a process going out of control, as opposed to occasionally producing defective parts in the course of its normal operation).
- The Markovian correlation creates a modeling problem since a Markovian description of the queueing process would require states that keep track of the quality of all items in the queue, as well as their order, leading to a computationally intractable situation. We circumvent this problem by using the following approximation which we call the *accelerated inspection model*.

All items are inspected with inspection (service) times assumed to be independent, exponentially distributed with inspection rate depending on the inspector phase: When the inspector is in phases $0, \ldots, k-1$ (corresponding to the full inspection mode) the inspection station is working with normal rate μ and inspections are considered perfect i.e. without errors. When the inspector is in phase k (corresponding to the sampling inspection mode) then the inspection station works at an accelerated rate $r\mu$ (where $r > 1$) and errors of type II, corresponding to defective items erroneously identified as nondefective can occur. The probability of occurrence of a type II error is $1 - \frac{1}{r}$. This model requires the introduction of an additional state, k_0, corresponding to the situation where a defective item is inspected with rate $r\mu$ and is erroneously accepted as nondefective. This state can be reached in one step only from state k with probability $q\left(1 - r^{-1}\right)$ or from itself with probability $q_0\left(1 - r^{-1}\right)$. From k_0 the inspector can make a transition to state k with probability p_0 or to state 0 with probability $q_0 r^{-1}$.

The system is thus described by a continuous time Markov chain with generator Q of the form (10) where A_i, $i = 0, 1, 2$, and B_0 are $(k+2) \times (k+2)$ matrices given by

$$
\begin{aligned}
B_0 &= -\lambda I, \quad A_0 = \lambda I \\
A_1 &= \begin{bmatrix} -(\lambda+\mu) & & & & & \\ & -(\lambda+\mu) & & & & \\ & & \ddots & & & \\ & & & -(\lambda+\mu) & & \\ & & & & -(\lambda+r\mu) & \\ & & & & & -(\lambda+r\mu) \end{bmatrix}, \\
A_2 &= \begin{bmatrix} \mu q_0 & \mu p_0 & & & \\ \mu q & & \mu p & & \\ \vdots & & & \ddots & \\ \mu q & & & \mu p & \\ \mu q & & & \mu r p & \mu r q\left(1-r^{-1}\right) \\ \mu q_0 & & & \mu r p_0 & \mu r q_0\left(1-r^{-1}\right) \end{bmatrix}
\end{aligned}
$$

The stability condition is $\pi A_0 e < \pi A_2 e$ where π is the vector of stationary probability of the matrix $A = A_0 + A_1 + A_2$

$$A = \begin{bmatrix} -\mu p_0 & \mu p_0 & & & & \\ \mu q & -\mu & \mu p & & & \\ \vdots & & \ddots & \ddots & & \\ \mu q & & & -\mu & \mu p & \\ \mu q & & & & -\mu r q & \mu r q\left(1 - r^{-1}\right) \\ \mu q_0 & & & & \mu r p_0 & -\mu\left(q_0 + r p_0\right) \end{bmatrix}$$

The solution of the system $\pi A = 0$ is

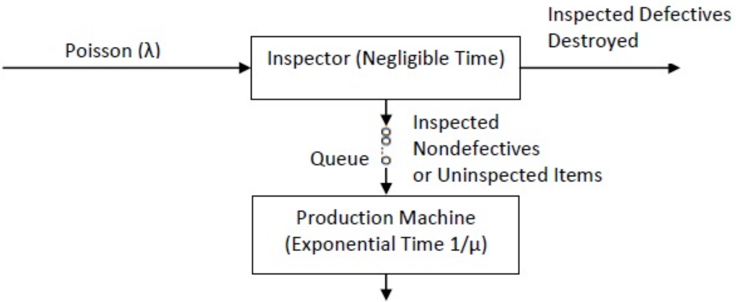
$$\begin{aligned} \pi_0 &= \frac{q}{q + p_0 - q\left(q_0 - q\right)\left(r - 1\right) p_0 p^{k-1}} \\ \pi_i &= \frac{q p_0 p^{i-1}}{q + p_0 - q\left(q_0 - q\right)\left(r - 1\right) p_0 p^{k-1}}, \qquad i = 1, \ldots, k-1 \\ \pi_k &= \frac{p_0 p^{k-1}\left(1 + p_0 (r - 1)\right)}{r\left(q + p_0 - q\left(q_0 - q\right)\left(r - 1\right) p_0 p^{k-1}\right)} \\ \pi_{k0} &= \frac{q p_0 p^{k-1} (r - 1)}{r\left(q + p_0 - q\left(q_0 - q\right)\left(r - 1\right) p_0 p^{k-1}\right)} \end{aligned}$$

The condition $\pi A_0 e < \pi A_2 e$ yields the stability condition

$$\lambda < \mu \left(1 + (r-1) \frac{p_0 p^{k-1} \left(1 + (p_0 + q)(r-1)\right)}{r \left(q + p_0 - q \left(q_0 - q\right)(r-1) p_0 p^{k-1}\right)} \right) \tag{15}$$

One can easily see that, when we adopt the Bernoulli arrivals for both models (diverted and accelerated) then for $0 < p < 1$, $\frac{1}{1 - p^k (1 - r^{-1})} < 1 + p^k (r-1)$ for all k and $r > 0$. Hence the stability region for the diverted steam model is smaller than that for the accelerated inspection model.

In this section we examine the modulating effect of the CSP inspector under the assumption that the inspection occurs at the final stage of a production process. The post–inspection stream replenishes a make–to–stock system which satisfies a Poisson demand with rate λ. This demand could arise from an intermediate stage of the production process. In such a situation one of the effects of the inspection process is that it increases the variability of the arrival process.



Here the generator Q of the process has the QBD form given by (10) where A_i, $i = 0, 1, 2$, and B_0 are $(k+2) \times (k+2)$ matrices given by

$$B_0 = \begin{bmatrix} -\lambda p_0 & & & & & \\ \lambda q & -\lambda & & & & \\ \vdots & & \ddots & & & \\ \lambda q & & & -\lambda & & \\ \lambda q r^{-1} & & & & -\lambda & \\ \lambda q_0 r^{-1} & & & & & -\lambda \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0 & \lambda p_0 & & & & \\ & 0 & \lambda p & & & \\ & & \ddots & \ddots & & \\ & & & 0 & \lambda p & \\ & & & & \lambda p & \lambda q \left(1 - r^{-1}\right) \\ & & & & \lambda p_0 & \lambda q_0 \left(1 - r^{-1}\right) \end{bmatrix}$$

$$
\begin{aligned}
A_1 &= \begin{bmatrix}
-(\lambda p_0 + \mu) & & & & & \\
\lambda q & -(\lambda + \mu) & & & & \\
\vdots & & \ddots & & & \\
\lambda q & & & -(\lambda + \mu) & & \\
\lambda q r^{-1} & & & & -(\lambda + \mu) & \\
\lambda q_0 r^{-1} & & & & & -(\lambda + \mu)
\end{bmatrix}, \\
A_2 &= \mu I_{(k+2)\times(k+2)}
\end{aligned}
$$

The stability condition for this system can once again be obtained from (14). The unique solution of $\pi A = 0$ is the one which also satisfies the normalization condition is

$$A = \begin{bmatrix} -\lambda p_0 & \lambda p_0 & & & & \\ \lambda q & -\lambda & \lambda p & & & \\ \vdots & & \ddots & \ddots & & \\ \lambda q & & & -\lambda & \lambda p & \\ \lambda q r^{-1} & & & & -\lambda q & \lambda q \left(1 - r^{-1}\right) \\ \lambda q_0 r^{-1} & & & & \lambda p_0 & -\lambda + \lambda q_0 \left(1 - r^{-1}\right) \end{bmatrix} \quad (16)$$

The solution of the system $\pi A = 0$ is

$$\begin{aligned} \pi_0 &= \frac{q}{q + p_0 + p_0 p^{k-1}(r-1)\left(1 + q - q_0\right)}, \\ \pi_i &= \frac{q p_0 p^{i-1}}{q + p_0 + p_0 p^{k-1}(r-1)\left(1 + q - q_0\right)}, \qquad i = 1, \ldots, k-1, \\ \pi_k &= \frac{\left(r - q_0\left(r-1\right)\right) p_0 p^{k-1}}{q + p_0 + p_0 p^{k-1}(r-1)\left(1 + q - q_0\right)}, \\ \pi_{k0} &= \frac{q\left(r-1\right) p_0 p^{k-1}}{q + p_0 + p_0 p^{k-1}(r-1)\left(1 + q - q_0\right)}. \end{aligned}$$

Thus (14) in this case gives the stability condition

$$\frac{p_0 + p_0 p^{k-1}(r-1)\,(1+q-q_0)}{q + p_0 + p_0 p^{k-1}(r-1)\,(1+q-q_0)} < \frac{\mu}{\lambda}.$$

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