Closed Finite Queueing Network Models with General Service Time Distributions

J. MacGregor Smith

Department of Mechanical and Industrial Engineering, Amherst MA, 01002, USA

May 28, 2015
Outline of Lecture Topics

A. Motivation

B. Background

C. Literature Review

D. Mathematical Model

E. Performance Algorithm

F. Experimental Results

G. Summary & Conclusions
Outline of Lecture Topics

- **A. Motivation**
- **B. Background**
- **C. Literature Review**
- **D. Mathematical Model**
- **E. Performance Algorithm**
- **F. Experimental Results**
- **G. Summary & Conclusions**
Outline of Lecture Topics

- A. Motivation
- B. Background
- C. Literature Review
  - D. Mathematical Model
  - E. Performance Algorithm
  - F. Experimental Results
  - G. Summary & Conclusions
Outline of Lecture Topics

- **A. Motivation**
- **B. Background**
- **C. Literature Review**
- **D. Mathematical Model**
- **E. Performance Algorithm**
- **F. Experimental Results**
- **G. Summary & Conclusions**

![Diagram of Input-Output systems with dimensions 40 ft, 80 ft, and 5 / 64 scale.]
Outline of Lecture Topics

A. **Motivation**
B. **Background**
C. **Literature Review**
D. **Mathematical Model**
E. **Performance Algorithm**
F. **Experimental Results**
G. **Summary & Conclusions**
Outline of Lecture Topics

- **A. Motivation**
- **B. Background**
- **C. Literature Review**
- **D. Mathematical Model**
- **E. Performance Algorithm**
- **F. Experimental Results**
- **G. Summary & Conclusions**
Outline of Lecture Topics

- **A. Motivation**
- **B. Background**
- **C. Literature Review**
- **D. Mathematical Model**
- **E. Performance Algorithm**
- **F. Experimental Results**
- **G. Summary & Conclusions**
Background

- **Assumptions**
  - Unpaced, asynchronous Flow line or FMS
  - Finite Buffers & Production Blocking
  - Closed Network Models, finite population, single-servers

- **Approximate Mean Value Analysis (MVA) Model**
  - Two-Moment General Service Model
  - Two-Moment Blocking Probability

- **Integrated Material Handling System**
Assumptions

- Unpaced, asynchronous Flow line or FMS
- Finite Buffers & Production Blocking
- Closed Network Models, finite population, single-servers

Approximate Mean Value Analysis (MVA) Model

- Two-Moment General Service Model
- Two-Moment Blocking Probability

Integrated Material Handling System
Background

- **Assumptions**
  - Unpaced, asynchronous Flow line or FMS
  - Finite Buffers & Production Blocking
  - Closed Network Models, finite population, single-servers

- **Approximate Mean Value Analysis (MVA) Model**
  - Two-Moment General Service Model
  - Two-Moment Blocking Probability

- **Integrated Material Handling System**
Background

Assumptions
- Unpaced, asynchronous Flow line or FMS
- Finite Buffers & Production Blocking
- Closed Network Models, finite population, single-servers

Approximate Mean Value Analysis (MVA) Model
- Two-Moment General Service Model
- Two-Moment Blocking Probability

Integrated Material Handling System
Background

Assumptions
- Unpaced, asynchronous Flow line or FMS
- Finite Buffers & Production Blocking
- Closed Network Models, finite population, single-servers

Approximate Mean Value Analysis (MVA) Model
- Two-Moment General Service Model
- Two-Moment Blocking Probability

Integrated Material Handling System
Background

- Assumptions
  - Unpaced, asynchronous Flow line or FMS
  - Finite Buffers & Production Blocking
  - Closed Network Models, finite population, single-servers

- Approximate Mean Value Analysis (MVA) Model
  - Two-Moment General Service Model
  - Two-Moment Blocking Probability

- Integrated Material Handling System
**Background**

- **Assumptions**
  - Unpaced, asynchronous Flow line or FMS
  - Finite Buffers & Production Blocking
  - Closed Network Models, finite population, single-servers

- **Approximate Mean Value Analysis (MVA) Model**
  - Two-Moment General Service Model
  - Two-Moment Blocking Probability

- **Integrated Material Handling System**
Background

- **Assumptions**
  - Unpaced, asynchronous Flow line or FMS
  - Finite Buffers & Production Blocking
  - Closed Network Models, finite population, single-servers

- **Approximate Mean Value Analysis (MVA) Model**
  - Two-Moment General Service Model
  - Two-Moment Blocking Probability

- **Integrated Material Handling System**
Literature Review

Infinite Buffers
- Marie’s Method
  - Marie, 1979
- Diffusion Approximation
  - Gaver and Shdler, 1973
  - Gelenbe, 1975
  - Tijms, 1992
  - Kimura, 1996
- Two-moment Approximations
  - Buzacott et.al., 1993
  - Curry and Feldman, 2008

Finite Buffers
- Marie’s Method
  - Akyildiz, 1988
  - Dayar and Meri, 2008
- Linear Programming
  - Matta and Chefson, 2005
  - Helber, et. al., 2011
- Open Network Approximation
  - Bouhchouch, 1996
  - Lagershausen, et.al., 2013
Basic Issues:

- Developing a closed network approximation for generally distributed finite blocking processes?
- Accounting for blocking from General distributions?
- Incorporating General probability distribution service information (moments and variability)?
- Creating an efficient running time algorithm?
Basic Issues:

- **Developing a closed network approximation for generally distributed finite blocking processes?**

- **Accounting for blocking from General distributions?**

- **Incorporating General probability distribution service information (moments and variability)?**

- **Creating an efficient running time algorithm?**
Closed General Finite Queueing Methodology

Basic Issues:

- Developing a closed network approximation for generally distributed finite blocking processes?

- Accounting for blocking from General distributions?

- Incorporating General probability distribution service information (moments and variability)?

- Creating an efficient running time algorithm?
Basic Issues:

- Developing a closed network approximation for generally distributed finite blocking processes?

- Accounting for blocking from General distributions?

- Incorporating General probability distribution service information (moments and variability)?

- Creating an efficient running time algorithm?
Queue Decomposition Mathematical Models

- Underlying logic behind Queue Decomposition idea:
  - $M/G/K/K$ queue acts as a holding node for the parts.
  - As the population increases, the congestion (blocking) increases as a function of the # of parts within the system.
  - Effective service rates decay as a function of the blocking in the system.
Queue Decomposition Mathematical Models

- **Underlying logic behind Queue Decomposition idea:**
  - $M/G/K/K$ queue acts as a holding node for the parts.
  - As the population increases, the congestion (blocking) increases as a function of the # of parts within the system.
  - Effective service rates decay as a function of the blocking in the system.
Queue Decomposition Mathematical Models

- Underlying logic behind Queue Decomposition idea:
  - $M/G/K/K$ queue acts as a holding node for the parts.
  - As the population increases, the congestion (blocking) increases as a function of the # of parts within the system.
  - Effective service rates decay as a function of the blocking in the system.
Queue Decomposition Mathematical Models

- Underlying logic behind Queue Decomposition idea:
  - *M/G/K/K* queue acts as a holding node for the parts.
  - As the population increases, the congestion (blocking) increases as a function of the # of parts within the system.
  - Effective service rates decay as a function of the blocking in the system.
M/G/c/c State Dependent Probability Distribution

For the $M/G/c/c$ probability distribution, we have for the idle time probability:

$$p_0 = \left( 1 + \sum_{n=1}^{C} \lambda^n \left( \prod_{j=1}^{n} j V_1 e^{-\left(\frac{j-1}{\beta}\right)^g L^{-1}} \right)^{-1} \right)^{-1}$$  \hspace{1cm} (1)

Finally, for the rest of the distribution:

$$p_n = n \leftrightarrow \lambda^n \left( \prod_{j=1}^{n} j V_1 e^{-\left(\frac{j-1}{\beta}\right)^g L^{-1}} \right)^{-1} \left( 1 + \sum_{n=1}^{C} \lambda^n \left( \prod_{j=1}^{n} j V_1 e^{-\left(\frac{j-1}{\beta}\right)^g L^{-1}} \right)^{-1} \right)^{-1}$$  \hspace{1cm} (2)
For the $M/G/c/c$ probability distribution, we have for the idle time probability:

$$p_0 = \left( 1 + \sum_{n=1}^{c} \lambda^n \left( \prod_{j=1}^{n} j V_1 e^{-\left( \frac{j-1}{\beta} \right)^g L^{-1}} \right)^{-1} \right)^{-1}$$  \hspace{1cm} (1)

Finally, for the rest of the distribution:

$$p_n = n \mapsto \lambda^n \left( \prod_{j=1}^{n} j V_1 e^{-\left( \frac{j-1}{\beta} \right)^g L^{-1}} \right)^{-1} \left( 1 + \sum_{n=1}^{c} \lambda^n \left( \prod_{j=1}^{n} j V_1 e^{-\left( \frac{j-1}{\beta} \right)^g L^{-1}} \right)^{-1} \right)$$  \hspace{1cm} (2)
For the \(M/G/c/c\) probability distribution, we have for the idle time probability:

\[
p_0 = \left( 1 + \sum_{n=1}^{\infty} \lambda^n \left( \prod_{j=1}^{n} j V_1 e^{-\left(\frac{j-1}{\beta}\right)^g L^{-1}} \right)^{-1} \right)^{-1}
\]

(1)

Finally, for the rest of the distribution:

\[
p_n = n \mapsto \lambda^n \left( \prod_{j=1}^{n} j V_1 e^{-\left(\frac{j-1}{\beta}\right)^g L^{-1}} \right)^{-1} \left( 1 + \sum_{n=1}^{\infty} \lambda^n \left( \prod_{j=1}^{n} j V_1 e^{-\left(\frac{j-1}{\beta}\right)^g L^{-1}} \right)^{-1} \right)^{-1}
\]

(2)
In the M/G/c/c state dependent model, the departure process (including both customers completing service and those that are lost) is a Poisson process at rate $\lambda$. 

**Proposition (Quasi-Reversibility Cheah and Smith, 94)**
Proposition (Upper Bound on Decomposition Throughput)

As $\mu_n \to \infty$, $\theta(N) \leq \theta(\infty)$ i.e. the throughput of the queue decomposition is bounded above by the throughput of an infinite capacity system.
Blocking Probability (Two moment estimation)

If one fixes the number of servers, one can solve for the blocking probability of the $M/M/1/K$ system.

$$ p_K = \frac{(1 - \rho) \rho^K}{1 - \rho^{K+1}} \Rightarrow K = \left\lceil \frac{\ln(p_K/(1 - \rho + p_K \rho))}{\ln(\rho)} \right\rceil $$  \hspace{1cm} (3)

$$ B = \left( \ln\left( \frac{p_K}{1 - \rho + p_K \rho} \right) - \ln(\rho) \right) \left( 2 + \sqrt{\frac{\rho}{es^2}} s^2 - \sqrt{\frac{\rho}{es^2}} \right) $$ \hspace{1cm} (4)

In the case of $c = 1$, the following expression is obtained for the blocking probability:

$$ p_K = \frac{\sqrt{\rho s^2 - \sqrt{\rho} + 2K}}{\rho^{2 + \sqrt{\rho s^2 - \sqrt{\rho}}}} \left( \rho - 1 \right) $$ \hspace{1cm} (5)
$P_K$ Comparisons $M/G/1/2 \ s^2 = \frac{1}{2}$

$P_K$ Comparisons $M/G/1/2 \ s^2 = 2$
The standard Equation 6 in the MVA for the expected delay time at a queue is based upon the PASTA property that

$$w_\ell(N) = \tau_\ell [1 + n_\ell(N - 1)]$$  \hspace{1cm} (6)

Accounting for the remaining service time which is a function of the utilization of the queue, the full service time of the number of customers in the queue, and the full service time of the arriving customer:

$$w_\ell(N) = \rho_\ell(N - 1) \frac{\tau_\ell(1 + s^2)}{2} + (n_\ell(N - 1) - \rho_\ell(N - 1))\tau_\ell + \tau_\ell$$  \hspace{1cm} (7)
Reiser and Lavenberg’s modified property of product-form networks to estimate the delay or residence time at the queue:

\[ w_\ell(N) = \rho_\ell(N - 1) \frac{\tau_\ell(1 + s^2)}{2} + (n_\ell(N - 1) - \rho_\ell(N - 1))\tau_\ell + \tau_\ell \] (8)

Little’s equation for product chains:

\[ \lambda_\ell(N) = \frac{N}{[\sum_{\ell=1}^{m} w_\ell(N)\alpha_\ell]} \] (9)

Little’s equation for queues:

\[ n_\ell(N) = \lambda_\ell(N)w_\ell(N) \] (10)
\[ \rho_\ell = \frac{n_\ell}{K_\ell} \approx \rho_\ell = \frac{\theta_\ell}{\mu_\ell} \]

- **[Step 1.0:]** Solve for \( \theta^U \) in the infinite buffer network with a given squared coefficient of variation \( s^2 \).
- **[Step 2.0:]** Find the bottleneck queues \( \ell^\beta \) in this topology with maximum \( \rho^\beta = \frac{\theta_\ell}{\mu_\ell} \).
- **[Step 3.0:]** Set up a finite buffer network with \( N, K_\ell, s^2_\ell \) and find the total utilization rate across the network:
  - **[Step 3.1:]** \( \rho_T = \sum_{\ell=1}^{M} \rho_\ell \).
  - **[Step 3.2:]** Set the lower bound velocity to:
    \[ V_1^\ell = V_1 \left( \frac{1 - \rho_\ell}{\rho_T} \right) \]

- This lower bound value will be useful in the general algorithm.
$V_1 = V_1 \exp^{-\rho s^2}$

$K := \text{capacity} = c + b$

Figure: Shifted Exponential Distribution
Throughput Buffer Envelope

Throughput versus $S^2$ Comparison

Figure: Upper and Lower Bounds $\theta$ vs. $s^2$
## Two-Stage Series Experiments

### Products

- $K_1 = 4, \mu_1 = 1$
- $K_2 = 4, \mu_2 = 1$

### Products

- $M/M/1$
- $M/G/4$
- $M/G/4$
- $M/M/1$

### Table

<table>
<thead>
<tr>
<th>$s_1^2$</th>
<th>$s_2^2$</th>
<th>$\theta(10)_e^a$</th>
<th>$\theta(10)_s^b$</th>
<th>$\theta(10)_a^c$</th>
<th>$\theta(10)_m^d$</th>
<th>% dev.</th>
<th>$\theta(10)_l^e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>0.50</td>
<td>1.9790</td>
<td>1.9780</td>
<td>1.9761</td>
<td>1.9522</td>
<td>1.35</td>
<td>1.892</td>
</tr>
<tr>
<td>0.50</td>
<td>1.00</td>
<td>1.9469</td>
<td>1.9460</td>
<td>1.9469</td>
<td>1.9281</td>
<td>0.50</td>
<td>1.838</td>
</tr>
<tr>
<td>0.50</td>
<td>4.00</td>
<td>1.7769</td>
<td>1.7916</td>
<td>1.7992</td>
<td>1.7897</td>
<td>0.72</td>
<td>1.572</td>
</tr>
<tr>
<td>0.50</td>
<td>9.00</td>
<td>1.6632</td>
<td>1.6609</td>
<td>1.6906</td>
<td>1.7231</td>
<td>3.60</td>
<td>1.318</td>
</tr>
<tr>
<td>1.00</td>
<td>0.50</td>
<td>1.9402</td>
<td>1.9388</td>
<td>1.9402</td>
<td>1.8593</td>
<td>4.17</td>
<td>1.836</td>
</tr>
<tr>
<td>1.00</td>
<td>1.00</td>
<td>1.9038</td>
<td>1.9026</td>
<td>1.9038</td>
<td>1.8712</td>
<td>1.71</td>
<td>1.780</td>
</tr>
<tr>
<td>1.00</td>
<td>4.00</td>
<td>1.7510</td>
<td>1.7602</td>
<td>1.7510</td>
<td>1.8113</td>
<td>3.44</td>
<td>1.528</td>
</tr>
<tr>
<td>1.00</td>
<td>9.00</td>
<td>1.6497</td>
<td>1.6409</td>
<td>1.6497</td>
<td>1.6700</td>
<td>1.23</td>
<td>1.292</td>
</tr>
</tbody>
</table>

- $^a$ Exact method
- $^b$ Simulation with Arena
- $^c$ Akyildiz’s method
- $^d$ M/G/c/c method
- $^e$ Lagershausen’s et.al. open network model
Throughput Curves $s^2 = \{0.50, 0.50\}; s^2 = \{1.00, 4.00\}$
### Three-Stage Experiments

![Three-Stage Diagram](image)

<table>
<thead>
<tr>
<th>$s^2$</th>
<th>$K_i$</th>
<th>$N^*$</th>
<th>$\theta(N)_e$</th>
<th>$\theta(N)_s$</th>
<th>$\theta(N)_m$</th>
<th>% dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>2</td>
<td>4</td>
<td>0.4393</td>
<td>0.4390</td>
<td>0.4552</td>
<td>2.94</td>
</tr>
<tr>
<td>1.00</td>
<td>3</td>
<td>6</td>
<td>0.4844</td>
<td>0.4842</td>
<td>0.5064</td>
<td>4.54</td>
</tr>
<tr>
<td>1.00</td>
<td>4</td>
<td>7</td>
<td>0.5116</td>
<td>0.5115</td>
<td>0.5234</td>
<td>2.31</td>
</tr>
<tr>
<td>1.00</td>
<td>5</td>
<td>9</td>
<td>0.5322</td>
<td>0.5321</td>
<td>0.5421</td>
<td>1.86</td>
</tr>
<tr>
<td>1.00</td>
<td>6</td>
<td>10</td>
<td>0.5463</td>
<td>0.5465</td>
<td>0.5403</td>
<td>1.10</td>
</tr>
<tr>
<td>0.64</td>
<td>2</td>
<td>4</td>
<td>0.4795</td>
<td>0.4782</td>
<td>0.4823</td>
<td>0.58</td>
</tr>
<tr>
<td>0.64</td>
<td>3</td>
<td>6</td>
<td>0.5233</td>
<td>0.5228</td>
<td>0.5326</td>
<td>1.78</td>
</tr>
<tr>
<td>0.64</td>
<td>4</td>
<td>7</td>
<td>0.5467</td>
<td>0.5463</td>
<td>0.5481</td>
<td>0.26</td>
</tr>
<tr>
<td>0.64</td>
<td>5</td>
<td>9</td>
<td>0.5629</td>
<td>0.5627</td>
<td>0.5500</td>
<td>2.20</td>
</tr>
<tr>
<td>0.64</td>
<td>6</td>
<td>11</td>
<td>0.5730</td>
<td>0.5728</td>
<td>0.5771</td>
<td>0.72</td>
</tr>
<tr>
<td>4.00</td>
<td>2</td>
<td>4</td>
<td>0.3502</td>
<td>0.3442</td>
<td>0.3209</td>
<td>8.37</td>
</tr>
<tr>
<td>4.00</td>
<td>3</td>
<td>6</td>
<td>0.3789</td>
<td>0.3763</td>
<td>0.3548</td>
<td>6.36</td>
</tr>
<tr>
<td>4.00</td>
<td>4</td>
<td>7</td>
<td>0.4000</td>
<td>0.4001</td>
<td>0.3950</td>
<td>1.25</td>
</tr>
<tr>
<td>4.00</td>
<td>5</td>
<td>9</td>
<td>0.4184</td>
<td>0.4207</td>
<td>0.4256</td>
<td>1.72</td>
</tr>
<tr>
<td>4.00</td>
<td>6</td>
<td>10</td>
<td>0.4335</td>
<td>0.4379</td>
<td>0.4384</td>
<td>1.13</td>
</tr>
</tbody>
</table>

- $a$ CTMC Markov Process method
- $b$ Simulation
- $c$ $M/G/c/c$ method
Three-Stage Split Experiments

\[ \alpha_{12} = 0.30 \]
\[ \alpha_{13} = 0.70 \]

<table>
<thead>
<tr>
<th>( s_1^2 )</th>
<th>( s_2^2 )</th>
<th>( s_3^2 )</th>
<th>( \theta(N) ) ( _s ) ( ^a )</th>
<th>( \theta(N) ) ( m ) ( ^b )</th>
<th>% dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>1</td>
<td>1</td>
<td>1.8751</td>
<td>1.8461</td>
<td>1.55</td>
</tr>
<tr>
<td>0.50</td>
<td>1</td>
<td>2</td>
<td>1.8089</td>
<td>1.8342</td>
<td>1.40</td>
</tr>
<tr>
<td>0.50</td>
<td>1</td>
<td>4</td>
<td>1.7256</td>
<td>1.8239</td>
<td>5.70</td>
</tr>
<tr>
<td>0.50</td>
<td>1</td>
<td>9</td>
<td>1.6022</td>
<td>1.6735</td>
<td>4.45</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1.8363</td>
<td>1.8073</td>
<td>1.58</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1.7743</td>
<td>1.7882</td>
<td>0.78</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
<td>1.6994</td>
<td>1.7962</td>
<td>5.70</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>9</td>
<td>1.5815</td>
<td>1.6406</td>
<td>3.74</td>
</tr>
</tbody>
</table>

\( ^a \) Simulation
\( ^b \) M/G/c/c method
# Four Stage Tandem Experiments

![Diagram of four stages](diagram.png)

<table>
<thead>
<tr>
<th>$s^2$</th>
<th>$K_i$</th>
<th>$N^*$</th>
<th>$\theta(N)_e^a$</th>
<th>$\theta(N)_s^b$</th>
<th>$\theta(N)_m^c$</th>
<th>% dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>2</td>
<td>6</td>
<td>0.4166</td>
<td>0.4206</td>
<td>0.4561</td>
<td>9.48</td>
</tr>
<tr>
<td>1.00</td>
<td>3</td>
<td>8</td>
<td>0.4652</td>
<td>0.4653</td>
<td>0.4956</td>
<td>6.53</td>
</tr>
<tr>
<td>1.00</td>
<td>4</td>
<td>10</td>
<td>0.4971</td>
<td>0.4959</td>
<td>0.5178</td>
<td>4.16</td>
</tr>
<tr>
<td>1.00</td>
<td>5</td>
<td>12</td>
<td>0.5196</td>
<td>0.5174</td>
<td>0.5354</td>
<td>3.04</td>
</tr>
<tr>
<td>1.00</td>
<td>6</td>
<td>14</td>
<td>0.5361</td>
<td>0.5339</td>
<td>0.5464</td>
<td>1.92</td>
</tr>
<tr>
<td>0.64</td>
<td>2</td>
<td>6</td>
<td>0.4617</td>
<td>0.4652</td>
<td>0.4801</td>
<td>3.99</td>
</tr>
<tr>
<td>0.64</td>
<td>3</td>
<td>8</td>
<td>0.5085</td>
<td>0.5096</td>
<td>0.5153</td>
<td>1.34</td>
</tr>
<tr>
<td>0.64</td>
<td>4</td>
<td>10</td>
<td>0.5365</td>
<td>0.5369</td>
<td>0.5390</td>
<td>0.47</td>
</tr>
<tr>
<td>0.64</td>
<td>5</td>
<td>12</td>
<td>0.5547</td>
<td>0.5548</td>
<td>0.5515</td>
<td>0.58</td>
</tr>
<tr>
<td>0.64</td>
<td>6</td>
<td>14</td>
<td>0.5671</td>
<td>0.5672</td>
<td>0.5680</td>
<td>0.16</td>
</tr>
<tr>
<td>4.00</td>
<td>2</td>
<td>5</td>
<td>0.3092</td>
<td>0.3078</td>
<td>0.3017</td>
<td>2.43</td>
</tr>
<tr>
<td>4.00</td>
<td>3</td>
<td>7</td>
<td>0.3408</td>
<td>0.3423</td>
<td>0.3397</td>
<td>0.32</td>
</tr>
<tr>
<td>4.00</td>
<td>4</td>
<td>9</td>
<td>0.3659</td>
<td>0.3692</td>
<td>0.3853</td>
<td>5.30</td>
</tr>
<tr>
<td>4.00</td>
<td>5</td>
<td>11</td>
<td>0.3865</td>
<td>0.3914</td>
<td>0.4107</td>
<td>6.26</td>
</tr>
<tr>
<td>4.00</td>
<td>6</td>
<td>13</td>
<td>0.4042</td>
<td>0.4106</td>
<td>0.4316</td>
<td>6.78</td>
</tr>
</tbody>
</table>

- $a$: CTMC Markov Process method
- $b$: Simulation
- $c$: $M/G/c/c$ method

**Figure:** 4-stage Comparison Results
Four-stage Split

\[ \alpha_{42} = 0.45 \]
\[ \alpha_{43} = 0.55 \]

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \theta(N)_s )</th>
<th>( \theta(N)_m )</th>
<th>% dev.</th>
<th>( W_s )</th>
<th>( W_m )</th>
<th>% dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.7097</td>
<td>0.7221</td>
<td>1.75</td>
<td>14.090</td>
<td>13.848</td>
<td>1.72</td>
</tr>
<tr>
<td>13</td>
<td>0.7337</td>
<td>0.7475</td>
<td>1.85</td>
<td>17.717</td>
<td>17.391</td>
<td>1.84</td>
</tr>
</tbody>
</table>
Five-Stage Split Topology

Figure: Five-stage Split-Merge Topology Network
Figure: Five Node Split Throughput Curve $s^2 = 1$
Figure: 5-stage Split Throughput Curve $s^2 = 1/2$
Figure: 5-stage Split Throughput Curve $s^2 = 2$
## Seven-Stage Balanced Tandem Line

### Figure: Seven stage Tandem Line Comparison

<table>
<thead>
<tr>
<th>#</th>
<th>$s^2$</th>
<th>$\theta(N)_s^a$</th>
<th>$\theta(N)_b^b$</th>
<th>%</th>
<th>$\theta(N)_\ell^c$</th>
<th>%</th>
<th>$\theta(N)_m^d$</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>0.7664</td>
<td>0.7486</td>
<td>2.3</td>
<td>0.7640</td>
<td>0.2</td>
<td>0.8063</td>
<td>5.21</td>
</tr>
<tr>
<td>2</td>
<td>0.50</td>
<td>0.8567</td>
<td>0.8464</td>
<td>1.2</td>
<td>0.8561</td>
<td>0.1</td>
<td>0.8455</td>
<td>1.31</td>
</tr>
<tr>
<td>3</td>
<td>2.00</td>
<td>0.6482</td>
<td>0.6282</td>
<td>3.1</td>
<td>0.6470</td>
<td>0.1</td>
<td>0.6929</td>
<td>6.90</td>
</tr>
<tr>
<td>4</td>
<td>5.00</td>
<td>0.4936</td>
<td>0.4570</td>
<td>7.4</td>
<td>0.4791</td>
<td>2.9</td>
<td>0.5570</td>
<td>12.84</td>
</tr>
</tbody>
</table>

- $^a$ Simulation
- $^b$ Bouhchouchm, Frein, and Dallery
- $^c$ Lagershausen’s et.al. Open Network model
- $^d$ $M/G/c/c$ model

---

### Table:

<table>
<thead>
<tr>
<th>Number</th>
<th>$s^2$</th>
<th>$\theta(N)_s$</th>
<th>$\theta(N)_b$</th>
<th>$\theta(N)_\ell$</th>
<th>$\theta(N)_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>0.7664</td>
<td>0.7486</td>
<td>0.7640</td>
<td>0.8063</td>
</tr>
<tr>
<td>2</td>
<td>0.50</td>
<td>0.8567</td>
<td>0.8464</td>
<td>0.8561</td>
<td>0.8455</td>
</tr>
<tr>
<td>3</td>
<td>2.00</td>
<td>0.6482</td>
<td>0.6282</td>
<td>0.6470</td>
<td>0.6929</td>
</tr>
<tr>
<td>4</td>
<td>5.00</td>
<td>0.4936</td>
<td>0.4570</td>
<td>0.4791</td>
<td>0.5570</td>
</tr>
</tbody>
</table>

---

### Notes:

- $s^2$ is the variance of the process.
- $\theta(N)$ represents the utilization of the system.
- $\ell$ indicates the effective utilization.
- $m$ denotes the maximum utilization.

---

### References:

- Simulation: Bouhchouchm, Frein, and Dallery
- Lagershausen’s et.al. Open Network model
- $M/G/c/c$ model
Unbalanced Seven-Stage Line Throughput Curves

Figure: Throughput Curve 7-stage $s^2 = 1, 1/2$
Ten Stage Topology

![Diagram of a ten-stage topology with service rates and buffer values.]

**Service Rates**

<table>
<thead>
<tr>
<th>Stage</th>
<th>Service Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>8</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>2</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>2</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>2.5</td>
</tr>
<tr>
<td>$\mu_5$</td>
<td>2.5</td>
</tr>
<tr>
<td>$\mu_6$</td>
<td>4</td>
</tr>
<tr>
<td>$\mu_7$</td>
<td>2.5</td>
</tr>
<tr>
<td>$\mu_8$</td>
<td>10</td>
</tr>
<tr>
<td>$\mu_9$</td>
<td>8</td>
</tr>
<tr>
<td>$\mu_{10}$</td>
<td>10</td>
</tr>
</tbody>
</table>

**Buffer Values**

<table>
<thead>
<tr>
<th>Stage</th>
<th>Buffer Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>6</td>
</tr>
<tr>
<td>$K_2$</td>
<td>7</td>
</tr>
<tr>
<td>$K_3$</td>
<td>7</td>
</tr>
<tr>
<td>$K_4$</td>
<td>6</td>
</tr>
<tr>
<td>$K_5$</td>
<td>5</td>
</tr>
<tr>
<td>$K_6$</td>
<td>4</td>
</tr>
<tr>
<td>$K_7$</td>
<td>8</td>
</tr>
<tr>
<td>$K_8$</td>
<td>6</td>
</tr>
<tr>
<td>$K_9$</td>
<td>6</td>
</tr>
<tr>
<td>$K_{10}$</td>
<td>5</td>
</tr>
</tbody>
</table>

**Table:** 10-stage Parameters
Figure: Throughput Curve 10-stage $s^2 = 1, 1/2$
Material Handling Systems

Table: 2-stage MHS model

<table>
<thead>
<tr>
<th>$s^2$: $\theta(N_\alpha)$</th>
<th>$\theta(N_s)$</th>
<th>$%$</th>
<th>$W_\alpha$</th>
<th>$W_s$</th>
<th>$%$</th>
<th>$Wip_\alpha$</th>
<th>$Wip_s$</th>
<th>$%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 : 0.872</td>
<td>0.846</td>
<td>3.07</td>
<td>10.317</td>
<td>10.382</td>
<td>0.63</td>
<td>0.872</td>
<td>0.846</td>
<td>3.07</td>
</tr>
<tr>
<td>1/2: 0.925</td>
<td>0.922</td>
<td>0.33</td>
<td>9.734</td>
<td>9.758</td>
<td>0.25</td>
<td>0.925</td>
<td>0.899</td>
<td>2.89</td>
</tr>
<tr>
<td>2 : 0.802</td>
<td>0.798</td>
<td>0.50</td>
<td>11.221</td>
<td>11.280</td>
<td>0.52</td>
<td>0.802</td>
<td>0.778</td>
<td>3.08</td>
</tr>
</tbody>
</table>
**Four-Stage MHS**

![Four-Stage MHS Diagram](image)

<table>
<thead>
<tr>
<th>$s^2 : \theta(N_\alpha)$</th>
<th>$\theta(N_s)$</th>
<th>%</th>
<th>$W_\alpha$</th>
<th>$W_s$</th>
<th>%</th>
<th>$W_{ip_\alpha}$</th>
<th>$W_{ip_s}$</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 : 0.825</td>
<td>0.801</td>
<td>3.00</td>
<td>20.605</td>
<td>21.227</td>
<td>2.93</td>
<td>3.614</td>
<td>3.300</td>
<td>9.52</td>
</tr>
<tr>
<td>1/2:0.868</td>
<td>0.879</td>
<td>1.25</td>
<td>19.585</td>
<td>19.331</td>
<td>1.31</td>
<td>3.573</td>
<td>3.633</td>
<td>1.65</td>
</tr>
<tr>
<td>2 : 0.764</td>
<td>0.702</td>
<td>8.83</td>
<td>22.252</td>
<td>24.222</td>
<td>8.13</td>
<td>3.674</td>
<td>3.666</td>
<td>0.22</td>
</tr>
</tbody>
</table>

**Table: 4-stage MHS model**
Figure: Throughput Curve $s^2 = 1$ Comparison

Figure: 4-stage MHS model
Six-Stage MHS

Table: 6-stage MHS model

<table>
<thead>
<tr>
<th>$s^2: \theta(N_\alpha)$</th>
<th>$\theta(N_s)$</th>
<th>%</th>
<th>$W_\alpha$</th>
<th>$W_s$</th>
<th>%</th>
<th>$Wip_\alpha$</th>
<th>$Wip_s$</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 : 0.817</td>
<td>0.783</td>
<td>4.34</td>
<td>31.836</td>
<td>33.203</td>
<td>4.12</td>
<td>3.772</td>
<td>3.400</td>
<td>10.94</td>
</tr>
<tr>
<td>1/2: 0.857</td>
<td>0.868</td>
<td>1.27</td>
<td>30.352</td>
<td>29.952</td>
<td>1.19</td>
<td>3.736</td>
<td>3.067</td>
<td>21.81</td>
</tr>
<tr>
<td>2: 0.759</td>
<td>0.673</td>
<td>12.78</td>
<td>34.255</td>
<td>38.633</td>
<td>11.33</td>
<td>3.824</td>
<td>3.065</td>
<td>24.76</td>
</tr>
</tbody>
</table>
Figure: Six-stage Split Topology
Six-Stage Split Throughput Curves

Throughput versus Population N Scv=1 Comparison

Throughput versus Population N Scv=1/2 Comparison

Figure: Throughput Curve 6-stage split $s^2 = 1, 1/2$
Multi-Chain MHS Layout Queueing Network
Multi-Chain MHS Layout Queueing Network

Input/Output

C1

80 ft

40 ft

C2

40 ft

K1

K2

K3

K4

K5

K6

K7

K8

K9

K10

59 / 64
There are a total of thirty-eight nodes with single-server and material handling conveyors.

Finite buffers of $K = 3$ at each station and $\mu = 2$ except at the input-output stations which are infinite server nodes.

Thus, this is an example of an Engset queueing network topology.

This is also a multi-chain system with two chains and varying populations. So this is a very complex closed queueing network topology.
Figure: Multi-Chain Layout Arena Simulation Model
Multi-Chain MHS Layout Throughput Curves

Figure: Layout Throughput Curve $s^2 = 1, 1/2$
Closed Finite Queueing Network Models

- Queue Decomposition concept
- Performance & Optimization Problems
- General Service Times
- Including the material handling system.

Open Questions & Extensions

- $V_1$ parameter refinements
- Optimization Models
  - $\{K_i, N, \mu_i, c_i\}$ & Layout Topologies
- Open Network Models
- General Multiple Servers

Generalized Engset Networks
Summary & Conclusions and Open Questions

- **Closed Finite Queueing Network Models**
  - Queue Decomposition concept
  - Performance & Optimization Problems
  - General Service Times
  - Including the material handling system.

- **Open Questions & Extensions**
  - $V_1$ parameter refinements
  - Optimization Models
    - $\{K_i, N, \mu_i, c_i\}$ & Layout Topologies
  - Open Network Models
  - General Multiple Servers

---

**Generalized Engset Networks**

![Diagram](image-url)