Customer Equilibrium Strategies in a Feed Forward Queueing Network with Join Topology

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Outline

Selling a Product under Monopoly - Demand Function

- p Price per unit
- $D(p)$ Demand (sales) under price p.
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Given a price p consumer i will buy a unit with probability $P(\Theta_i > p) = 1 - F(p)$. Thus,

$$D(p) = N(1 - F(p))$$

decreasing in p.

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- Perception of better quality
- Externality: "If price goes up I will be one of the few who will buy the product" (prestige effect).

Related Phenomena Under Queueing Effects

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Call Center - Customer Support

- 2 phone numbers for support
- First number free, second with charge per minute
- Same type of support
- Why call the paid number?

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Economic/Game theoretic problems with a queueing underlying model.

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Nash Equilibrium Strategies

A strategy p_e is a Symmetric Nash Equilibrium (SNE) if Assuming all customers follow p_e, no individual customer can improve his net profit by changing his own p value.

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- p_e is a SNE strategy if it is a best response to itself : $p^*(p_e) = p_e$.

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- It is always true that $\lambda_e \leq \mu - \frac{C}{R}$, i.e., the system self adjusts to an unused capacity of at least $\frac{C}{R}$.
- This happens because customers are individual decision makers and are not forced to join if high congestion is anticipated.

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$$S(p^0) = \max_{0 \leq p \leq 1} S(p)$$

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- $S(p)$ is concave: to maximize, set $S'(p) = 0$.

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In summary

	p^0	$\lambda^0 = \Lambda p^0$
$\mu - \sqrt{\frac{C\mu}{R}} < \Lambda$	$\frac{\mu - \sqrt{\frac{C\mu}{R}}}{\Lambda}$	$\mu - \sqrt{\frac{C\mu}{R}}$
$\Lambda \leq \mu - \sqrt{\frac{C\mu}{R}}$	1	Λ

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5. Negative externalities: $-CW'(p^0) < 0$.

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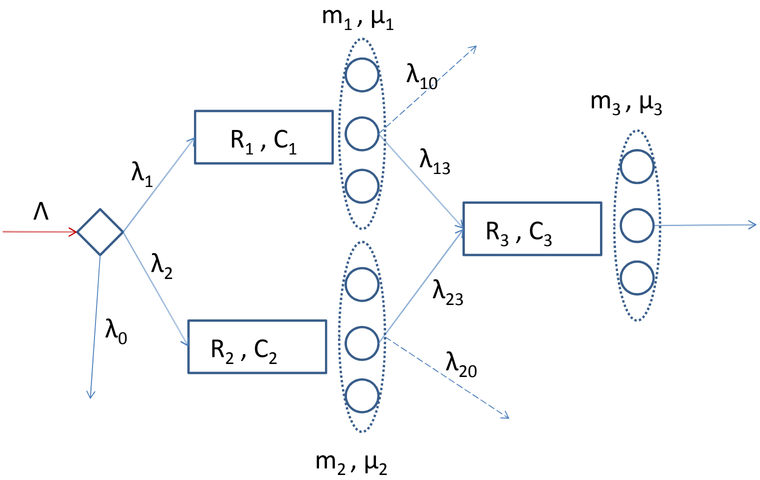
6. Many more.

Strategic Customers in a Feed Forward Queueing Network

Problem Setting

- A two-stage open-Jackson network
- Two stations in the first stage, lead to a single station in the second stage
- Customers act individually
  - They arrive at the first stage
  - Decide which, if any, first-stage queue to join
  - Second-stage service may be mandatory or optional
  - Maximize expected net benefit = Service reward - Delay cost
- Unobservable System
- Questions
  - Equilibrium join/routing strategy.
  - Socially optimal strategy.

Model Description



Reneging vs No Reneging

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- Case 1: No Reneging
  - Second stage service is mandatory.
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  - Appropriate when second stage is essential to service completion or customers cannot depart in the middle because of physical layout of facility.

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- Case 2: Reneging
  - Second stage service optional.
  - Some customers may choose not to receive it.
- How does the nature of the second stage (mandatory vs optional) affect the formation of equilibrium strategies?

No Reneging (Model N)

- Space of mixed strategies: $X_N = \{\mathbf{x}_N : x_1, x_2 \geq 0, x_1 + x_2 \leq 1\}$, x_i=Probability of joining queue i.
- $W_i(\Lambda x_i)$ =expected total delay in steady state in queue i, for input rate Λx_i, $i = 1, 2, 3$.
- $u_i(x_i) = R_i - C_i W_i(\Lambda x_i)$ =expected net benefit of a customer who joins queue i if all other customers join queue i with probability x_i.
- Optimal Response problem to strategy $\mathbf{x}_N = (x_1, x_2)$:

$$F_N(\mathbf{x}_N) = \max\{B_{N1}(x_1, x_2)y_1 + B_{N2}(x_1, x_2)y_2, \quad y_1, y_2 \geq 0, y_1 + y_2 \leq 1\},$$

where $B_{N1}(x_1, x_2) = u_1(x_1) + u_3(x_1 + x_2)$, $B_{N2}(x_1, x_2) = u_2(x_2) + u_3(x_1 + x_2)$.

Optimal Response Function

Given that all customers follow strategy $\mathbf{x}_N = (x_1, x_2)$, the tagged customer's optimal response strategy $\mathbf{y}_N^*(\mathbf{x}_N$ is

$$\mathbf{y}_N^*(\mathbf{x}_N) = \begin{cases} (0,0), & \text{if } B_{N1}(x_1,x_2) < 0, B_{N2}(x_1,x_2) < 0, \\ (y_1,0), & \text{if } B_{N1}(x_1,x_2) = 0, B_{N2}(x_1,x_2) < 0, \\ (0,y_2), & \text{if } B_{N1}(x_1,x_2) < 0, B_{N2}(x_1,x_2) = 0, \\ (y_1,y_2), & \text{if } B_{N1}(x_1,x_2) = B_{N2}(x_1,x_2) = 0, \\ (1,0), & \text{if } B_{N1}(x_1,x_2) > B_{N2}(x_1,x_2), B_{N1}(x_1,x_2) > 0, \\ (0,1), & \text{if } B_{N1}(x_1,x_2) < B_{N2}(x_1,x_2), B_{N2}(x_1,x_2) > 0, \\ (y_1, 1-y_1), & \text{if } B_{N1}(x_1,x_2) = B_{N2}(x_1,x_2) > 0 \end{cases}$$

Properties of Equilibrium Strategy

Lemma

Let $\mathbf{x}_N^e = (x_1^e, x_2^e)$ *be any equilibrium strategy for model N. Then the following hold:*

(1) *If* $B_{N1}(0,0) \leq 0$, *then* $x_1^e = 0$.
(2) *If* $B_{N2}(0,0) \leq 0$, *then* $x_2^e = 0$.
(3) *If* $B_{N1}(1,0) \geq 0$ *or* $B_{N2}(0,1) \geq 0$, *then* $x_1^e + x_2^e = 1$.

Based on this lemma and extending the analysis of the single stage unobservable queue we obtain

Theorem

There exists a unique symmetric equilibrium strategy $\mathbf{x}_N^e$ *in model N.*

The specific form of $\mathbf{x}_N^e$ is obtained by considering several cases.

Reneging (Model R)

- Space of mixed strategies:
  $X_R = \{\mathbf{x}_R : x_1, x_2, x_{13}, x_{23} \geq 0, x_1 + x_2 \leq 1, x_{i3} \leq x_i, i = 1, 2\}$
- Optimal Response problem to strategy $\mathbf{x}_R = (x_1, x_2, x_{13}, x_{23})$:

$$F_R(\mathbf{x}_R) = \max\{u_1(x_1)y_1 + u_2(x_2)y_2 + u_3(x_{13} + x_{23})(y_{13} + y_{23}) : \\ y_1, y_2, y_{13}, y_{23} \geq 0, y_1 + y_2 \leq 1, y_{13} \leq y_1, y_{23} \leq y_2\},$$

- The problem seems more complicated than the N case, because:
  1. There are 4 variables in a mixed strategy.
  2. There are multiple equilibria in joining the second queue. This is a problem because in principle a customer does not know which of them will actually occur, when deciding what to do at the first stage.
  3. Fortunately, it can be shown that all the expected benefit from joining the second queue is always the same, regardless of which equilibrium strategy is followed.
  4. We also have the following property: Under any equilibrium strategy

$$u_3(x_{13} + x_{23}) = u_3(x_1 + x_2)^+.$$

Reduction to the No Reneging Model

The previous properties allow for the problem to be reduced to one without reneging, and appropriately transformed benefit function.

Lemma

If x_1, x_2 are part of an equilibrium strategy in problem R, then they must be the optimal solution to the following optimal response problem

$$F_R(x_1, x_2) = \max\{B_{R1}(x_1, x_2)y_1 + B_{R2}(x_1, x_2)y_2, \quad y_1, y_2 \geq 0, y_1 + y_2 \leq 1\},$$

where $B_{R1}(x_1, x_2) = u_1(x_1) + u_3(x_1 + x_2)^+$, $B_{R2}(\mathbf{x}_R) = u_2(x_2) + u_3(x_1, x_2)^+$.

We thus obtain a similar existence and uniqueness result.

Theorem

In model R there generally exist infinite symmetric equilibrium strategies $\mathbf{x}_R^e = (x_1, x_2, x_{13}, x_{23})$*, such that the proportion of customers joining the queues* x_1, x_2, x_3 *are uniquely defined and* x_{13}, x_{23} *are any pair of probabilities satisfying* $x_{13} + x_{23} = x_3$.

Numerical Results

Service Rates $\mu_1 = 3, \mu_2 = 4, \mu_3 = 3$
Service Rewards $R_1 = 30, R_2 = 20, R_3 = 25$
Delay cost rates $C_1 = 6, C_2 = 5, C_3 = 8.$

	$\mathbf{x}_N^e$			$\mathbf{x}_R^e$			
Λ	x_1	x_2	$x_1 + x_2$	x_1	x_2	$x_1 + x_2$	$x_{13} + x_{23}$
2.5	0.987	0.013	1.000	0.987	0.013	1.000	1.000
3.0	0.824	0.115	0.939	0.825	0.175	1.000	0.893
3.5	0.706	0.098	0.805	0.710	0.290	1.000	0.766
4.0	0.618	0.086	0.704	0.625	0.375	1.000	0.670
4.5	0.549	0.077	0.626	0.560	0.440	1.000	0.596
5.0	0.494	0.069	0.563	0.509	0.491	1.000	0.536
5.5	0.449	0.063	0.512	0.471	0.529	1.000	0.487
6.0	0.412	0.057	0.469	0.443	0.557	1.000	0.447
6.5	0.380	0.053	0.433	0.428	0.572	1.000	0.412
7.0	0.353	0.049	0.402	0.400	0.536	0.936	0.383

Social Optimization Problem

- A central decision maker makes state independent acceptance/rejection and routing decisions x_1, x_2, x_{13}, x_{23} so as to maximize the total expected benefit of all customers per unit time.
- Negative Externalities: $x_i^* \leq x_i^e, i = 1, 2, 3$.

Extensions

- More general network structures - recursive equations for equilibria.
- Pricing.
- Capacity Planning.
- Observable Models (??)

Literature Review

- Equilibrium Models in Queueing Systems
  - Naor (1969), Edelson and Hildebrand(1974)
  - Hassin and Haviv (2003)
- Pricing in Networks
  - Kelly (1998)
  - Courcoubetis and Weber (2003)
  - Altman et al. (2006)
  - Zachariadis and Baria (2007)