NUMERICAL ALGORITHMS FOR TRANSIENT ANALYSIS OF FLUID QUEUES

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Abstract. We consider the continuous model of a two machines with finite intermediary buffer production line. The behavior of the machines is controlled by finite homogeneous Markov chains. We study a numerical algorithm for the integration of the system of partial differential equations describing the transient behavior. Stochastic fluid models are used for performance evaluation of high-speed computer networks and manufacturing systems. Solutions to the transient behavior of such systems were provided by previous works in terms of Laplace transform of the solution or by explicit formula in the case of an infinite buffer. Our numerical algorithm provides a fast solution with a good precision under weak assumptions on the initial conditions.

Keywords: Fluid flow model, Transient behavior, Numerical algorithms

1 Introduction

A stochastic fluid model is a system describing the behavior of a fluid flow in a storage device. The input and output are controlled by stochastic processes. Such models are used to approximate discrete queuing models for performance evaluation of computer networks, transportation systems or manufacturing systems. A large number of papers was devoted to the analysis of fluid models with Markov chains modulated input and output process in various application contexts. Most of the papers deal with stationary analysis of fluid models [1, 10, 4, 5, 13]. A good review of the results can be found in [9, 3].

Concerning the transient analysis of fluid models a number of papers uses Laplace transform methods for fluid models driven by Markov chains [8, 11]. The most recent advance is the Markov modulated input rate model studied by Tanaka et al. in [17] which extends previous results by Ren and Kobayashi [8, 11]. A closed form solution based was obtained by Sericola [12] for a infinite buffer fluid model controlled by a homogeneous Markov process.

In the manufacturing systems domain Hongler has developed a "boundary-free" model [7] for the analysis of the throughput variability and distribution of the first busy period of a tandem model. Further results were presented in [2]. In this paper we consider a finite buffer fluid model with input and output rates driven by finite homogeneous Markov chains. We use this system to model a two machines production system with unreliable machines and operation dependent failures. For this model we develop a numerical integration method for the evolution equations. The method is based on a sequential integration of the linear and the wave parts of the evolution equations followed by the application of the boundary conditions. We prove, for compatible initial conditions, that our algorithm provides a solution which is a probability distribution.

2 Model description

We consider a continuous flow production system with finite capacity intermediary buffer C. For convenience we consider the simple two state machines model with operations dependent failures assumption. The global systems behavior can be viewed as a fluid buffer with input and output rates controlled by a Markov chain $\{Y_t, t \geq 0\}$ with infinitesimal generator M, initial probability distribution $\vec{\alpha}$ and stationary probability vector $\vec{w}$. The buffer level at time t is denoted by X_t. Let v_i be the input rate and u_i the output rate when the Markov process Y_y is in state i. The the *drift* associated to the state i is $d_i = v_i - u_i$. We call the matrix $D = diag\{\vec{d}\}$ the drift matrix.

Let $\pi_i(t,x) = Pr[Y_t = i, X_t \leq x]$. The following partial differential equations system (see [10] or [12]) characterizes the system evolution:

$$\frac{\partial \vec{\pi}(t,x)}{\partial t} = -\frac{\partial \vec{\pi}(t,x)}{\partial x} D + \vec{\pi}(t,x) M. \tag{1}$$

Turning back to the two machines initial model suppose that each two state machine is characterized by the infinitesimal generator and rate vector $(Q_1, \vec{v}_1)$ respectively $(Q_2, \vec{v}_2)$:

$$Q_i = \begin{bmatrix} -\lambda_i & \lambda_i \\ \mu_i & -\mu_i \end{bmatrix}, \vec{v}_i = (c_i, 0) \text{ for } i = 1, 2, \tag{2}$$

where λ_i are the failure rates, and μ_i are repairing rates. It follows that the *Markovian drift process* ([10, 12]) is defined by the Markovian generator $M = Q_1 \oplus Q_2$, and the drift vector $\vec{d} = \vec{v}_1 \otimes \vec{1} - \vec{v}_2 \otimes \vec{1}$. Explicitly, with Q_1 and Q_2 given by (2), M and $\vec{d}$ given by:

$$\begin{aligned} M &= \begin{bmatrix} -(\lambda_1 + \lambda_2) & \lambda_2 & \lambda_1 & 0 \\ \mu_2 & -(\lambda_1 + \mu_2) & 0 & \lambda_1 \\ \mu_1 & 0 & -(\mu_1 + \lambda_2) & \lambda_2 \\ 0 & \mu_1 & \mu_2 & -(\mu_1 + \mu_2) \end{bmatrix} \\ \vec{d} &= (c_1 - c_2, c_1, -c_2, 0) \end{aligned} \tag{3}$$

The characterization provided by equation (1) does not take into account the boundary conditions, *i.e.* the behavior of the system when the buffer is empty or full.

The boundary conditions for the stationary regime ($t \to \infty$) where developed for exemple in [6, 4]. We well shortly remaind those conditions written in matrix form and then we will apply them to the transient regime.

Let $\vec{\pi}(x) = \lim_{t \to \infty} \vec{\pi}(t,x)$. It is known that the stationary solution of the two machine production system is of the form

$$\vec{\pi}(x) = \vec{P}(0)\delta_0 + \vec{\pi}_c(x) + \vec{P}(C)\delta_C,$$

where $\vec{P}_0$ and $\vec{P}(C)$ are the jump probability of the boundary states corresponding to the events $\{X = 0\} and \{X = C\}$.

- When the buffer is empty in the state i, if the drift d_i is positive that means that the input rate exceeds the output rate and then the buffer level instantaneously increase. That means that the probability to have the buffer empty in a positive drift state is zero: $P_i(0) = 0$ if $d_i > 0$

- When the buffer is full in the state i, if the drift d_i is negative, the output rate will exceed the input rate and the buffer instantaneously begins to empty. It follow that the probability $\pi_i(C)$ is equal to the probability of the state i. Then $P_i(C) = 0$ if $d_i < 0$.

Supplementary conditions on the boundary states comes from the operations dependent failures assumption.

- When the buffer is empty in the state i, if the drift d_i is negative that means that second machine is starved by the first one. Then the failure rate of the second machine is reduced accordingly by a factor $\vec{v}_1(i)/\vec{v}_2(i)$
- When the buffer is full in the state i, if the drift d_i is positive, the first machine is blocked by the second one and its failure rate is reduced by the coefficient $\vec{v}_2(i)/\vec{v}_1(i)$.

Then we can summarize the stationary boundary conditions as:

$$\begin{aligned} P_i(C) &= 0 \text{ if } d_i < 0, \\ P_i(0) &= 0 \text{ if } d_i > 0, \\ \dot{\vec{\pi}}_c(C)D &= -\vec{P}(C)N_C, \\ \dot{\vec{\pi}}_c(0) &= -(\vec{\pi}_c(0) + \vec{P}(0))N_0, \end{aligned}$$

where, for the two states machines production system the matrices N_0 and N_C are given by:

$$\begin{aligned} N_C &= \begin{bmatrix} -(a_1\lambda_1+\lambda_2) & \lambda_2 & a_1\lambda_1 & 0 \\ \mu_2 & -\mu_2 & 0 & 0 \\ \mu_1 & 0 & -(\mu_1+\lambda_2) & \lambda_2 \\ 0 & \mu_1 & \mu_2 & -(\mu_1+\mu_2) \end{bmatrix}, \\ N_0 &= \begin{bmatrix} -(\lambda_1+a_2\lambda_2) & a_2\lambda_2 & \lambda_1 & 0 \\ \mu_2 & -(\lambda_1+\mu_2) & 0 & \lambda_1 \\ \mu_1 & 0 & -\mu_1 & 0 \\ 0 & \mu_1 & \mu_2 & -(\mu_1+\mu_2) \end{bmatrix}, \end{aligned}$$

where $a_1 = \min(1, c_2/c_1)$ and $a_2 = \min(1, c_1/c_2)$.

For the rest of the paper we let $\vec{\pi}(t,h)$ denote the probability distribution function except the jump at C, *i.e.*:

$$\vec{\pi}(t,x) = P(t,0) + \vec{\pi}_c(t,x).$$

This convention is used to simplify the notations and, on another hand, is relatively natural, as:

$$Pr\{Y-t=i, X_t \leq x \| X_t \neq C\} = P_i(t,0) + \pi_{ic}(t,x).$$

One can easily see that the reasoning used to obtain the stationary boundary conditions easily extends to the transient regime. Then the following time-dependent boundary conditions hold:

$$\begin{aligned} P_i(t,C) &= 0 \text{ if } d_i < 0, \\ \pi_i(t,0) &= 0 \text{ if } d_i > 0, \\ \left.\frac{\partial \vec{\pi}(t,x)}{\partial x}\right|_{x=C} &= -\vec{P}(t,C)N_C, \\ \left.\frac{\partial \vec{\pi}(t,x)}{\partial x}\right|_{x=0} D &= -\vec{\pi}(t,0)N_0, \end{aligned} \tag{4}$$

The system of partial differential equations is completely specified by the initial conditions. We suppose that the initial level of the buffer is $x_0 \leq C$. Then the initial condition is:

$$\vec{\pi}(0,x) = \begin{cases} 0, & x < x_0 \\ \vec{\alpha}, & x \geq x_0 \end{cases} \qquad (5)$$

Suppose that there are n states of the Markov chain Y_t, $n_+ < n$ of positive drift and $n_- \leq n - n_+$ of negative drift. In [12] it was proved that, in the case of an infinite buffer and $x_0 = 0$, the probability distribution of the buffer level presents $n_+ + 1$ jumps in the point $d_i t$, for all i such that $d_i > 0$, as the buffer level takes values in the interval $[0, max(d_i)t]$. It will also be the case for a finite buffer with $x_0 > 0$. Moreover there are $n_- + 1$ supplementary jumps due to the negative drifts. Using the formula from [12] the values of the jumps (for $x_0 = 0$ is:

$$Pr\{Y_t = i, X_t = d_j t\} = \vec{\alpha}_{B_+} e^{M_{B_+B_+}t} e_i, \qquad (6)$$

where B_+ is the set of states with positive drifts, $M_{B_+B_+}$ is the sub-generator of M restricted to the transitions inside the set B_+ and $\vec{\alpha}_{B_+}$ is the restriction of $\vec{\alpha}$ to the states in B_+ and e_i is the i^th unit vector of adequate dimension. A formula similar to (6) can be deduced for the jumps induced by negative drift states.

A closer look at the formula (6) shows that the jumps will appear if the initial conditions are not compatible with the boundary conditions. Indeed, boundary conditions specifies that, for positive drift states, the probability of the event "buffer empty" is zero. But if $\vec{\alpha}_{B_+} > 0$ that means that in the initial state the probability to find the buffer empty in a positive drift state is non-zero.

As our algorithm provides a continuous and continuously differentiable solution, and then it cannot provide jumps, we will ask the initial conditions (5) to be compatible with the boundary conditions (4).

On the other hand, again in [12], was show that if even the value of the jump in the point $d_i t$ is zero, the solution is not differentiable in this points. We will then consider the best continuously differentiable approximation of the real solution for the convergence of our algorithm.

In the following we will develop an integration algorithm for the system (1) with initial condition (5) and boundary conditions (4).

3 The integration algorithm

The numerical resolution of the system (1) is more difficult than the integration of the usual partial differential equations. There is a supplementary implicit constraint which is that the solution $\vec{\pi}(t,x)$ is a probability distribution function for any t. Practically that means that the numerical method must provide a monotonic positive solution at every t. This difficulty is quit usual in the numerical analysis of Markov chains. Some papers where devoted to the study of specialized algorithms for Markov chain solutions [16, 14, 15]. We will show that our algorithm provides the required properties of the numerical solution.

The integration algorithm is based on the superposition method which is applied to the system (1). We decompose the system (1) in two systems:

$$\frac{\partial \vec{p}(t,x)}{\partial t} = \vec{p}(t,x)M, \text{ linear part} \qquad (7)$$

$$\frac{\partial \vec{q}(t,x)}{\partial t} = -\frac{\partial \vec{q}(t,x)}{\partial x} D, \text{ wave part} \tag{8}$$

We will define three transformations corresponding to the discret form of the solutions of (7) and (8) and to the boundary conditions. We consider that time and space increments Δt respectively Δx are defined. In the following, for the sake of simplicity, if there is no possible confusion, we will write k instead of $k\Delta t$ for the time at step k and j instead of $j\Delta x$ for the discretized level of the buffer $j\Delta x$.

3.1 Linear transform

The general solution of (7) is of the form: $\vec{p}(t,x) = \vec{p}(0,x)e^{Mt}$. The linear transformation correspond to the discrete form of the exponential solution for a time increment Δt:

$$\Psi^L(\vec{p}(k,x)) \stackrel{\text{def}}{=} \vec{p}(k+1,x) = \vec{p}(k+1,x)e^{M\Delta t} \tag{9}$$

3.2 Wave transform

The system (8) is a pseudo-wave equation systems. Then the discrete form of the solution which defines the wave transformation is:

$$\Psi^W(q_i(k,x)) \stackrel{\text{def}}{=} q_i(k+1,x) = q_i(k+1, x - d_i\Delta t), i = 1 \ldots n \tag{10}$$

3.3 Boundary conditions transform

Eventually, the discret form of the boundary conditions is used to define the transformation:

$$\Psi^B(\pi_i(k,x)) \stackrel{def}{=} \begin{cases} P_i(k+1,C) = 0 & \text{if } d_i < 0 \\ \pi_i(k+1,0) = 0, & \text{if } d_i > 0 \\ \text{solve} & \\ \begin{cases} \vec{\pi}(k+1,m)D + \Delta x \vec{P}(k+1,C)N_C = \vec{\pi}(k+1,m-1)D \\ \vec{\pi}(k+1,0)(D - \Delta x N_0) = \vec{\pi}(k+1,1)D \\ \|\vec{\pi}(k+1,m)\|_1 + \|\vec{P}(k+1,C)\|_1 = 1 \end{cases} & \\ \text{for } \vec{\pi}(k+1,m), \vec{P}(k+1,C), \vec{\pi}(k+1,0) & \end{cases} \tag{11}$$

The linear system appearing in (11) is the transient discretized matrix form of the classical boundary equations as in [6]. We have used the development:

$$\frac{\partial \vec{\pi}(k,j)}{\partial x} \approx \frac{\vec{\pi}(k,j) - \vec{\pi}(k,j-1)}{\Delta x}$$

We can state now the integration method.

Theorem 1 *The system system (1) with initial condition (5) compatible with the boundary conditions (4) can be solved by the integration scheme:*

$$\vec{\pi}(k+1,x) = \Psi^B\left(\Psi^W\left(\Psi^L\left(\vec{\pi}\left(k,x\right)\right)\right)\right), \tag{12}$$

with

$$\Delta t \leq \frac{\Delta x}{\max\limits_{i=1\dots n} |d_i|}, \tag{13}$$

where Δx is the chosen uniform spacing of the $[0, C]$ interval. The solution converges to the best continuously differentiable approximation of the real solution.

In order to prove this result we will show that the solution provided by the integration scheme (12) is positive monotonic and converges to the solution of the problem (1).

The following equivalent expression of the integration scheme is used for the proof of the Theorem 1.

Lemma 2 *The integration scheme (12) is equivalent to the explicit difference equations:*

$$\begin{array}{rcl}
\pi_i(k+1,j) & = & \left\{ \begin{array}{cl} \vec{\pi}(k,j)\Theta_i & d_i = 0, j = 0\dots m \\ (1-d_i r)\vec{\pi}(k,j)\Theta_i + d_i r \vec{\pi}(k,j-1)\Theta_i & d_i > 0, j = 1\dots m \\ 0 & d_i > 0, j = 0 \\ (1+d_i r)\vec{\pi}(k,j)\Theta_i - d_i r \vec{\pi}(k,j+1)\Theta_i & d_i < 0, j = 0\dots m-1 \end{array} \right. \\
P_i(k+1,C) & = & 0 \text{ if } d_i < 0 \\
solve & & \left\{ \begin{array}{l} \vec{\pi}(k+1,m)D + \Delta x \vec{P}(k+1,C)N_C = \vec{\pi}(k+1,m-1)D \\ \vec{\pi}(k+1,0)(D - \Delta x N_0) = \vec{\pi}(k+1,1)D \\ \|\vec{\pi}(k+1,m)\|_1 + \|\vec{P}(k+1,C)\|_1 = 1 \end{array} \right. \\
 & & for \left\{ \begin{array}{ll} \vec{\pi}(k+1,m), & \\ \vec{P}(k+1,C), & d_i \geq 0 \\ \vec{\pi}(k+1,0), & d_i \leq 0 \end{array} \right.
\end{array} \tag{14}$$

where Θ_i is the i^{th} column of the matrix exponential $e^{M\Delta t}$, $r = \frac{\Delta t}{\Delta x}$, and $m = \frac{C}{\Delta x}$.

Proof. In view of the definition (11) of Ψ^B is obvious for the cases $d_i > 0, j = 0$ and $d_i < 0, j = m$.

Case $d_i = 0$. In this case Ψ^W is the identity and Ψ^B is not defined. From the definition (9) of Ψ^L the expression in (14) is direct.

Case $d_i > 0, j > 0$. We apply a first-order interpolation development to the equation (10).

$$\begin{array}{rcl} q_i(k+1, j\Delta x) & = & q_i(k+1, j\Delta x - d_i \Delta t) = \\ & = & (1 - d_i r) q_i(k, j\Delta x) + d_i r q_i(k, (j-1)\Delta x). \end{array} \tag{15}$$

Using the expression (9) in (15), the expression in (14) follows.

Case $d_i < 0, j < m$. The proof is similar to the previous case.

The boundary conditions resolution follows directly from the definition of Ψ^B. $\square$

Corollary 3 ***(Proof of the positivity and monotonicity of $\vec{\pi}(k\Delta t, j\Delta x)$).*** *The condition (13) is sufficient to insure the monotonicity of the solution provided by (12)*

The condition (13) is equivalent to the condition:

$$|d_i|r < 1, \forall i. \tag{16}$$

Suppose that $\vec{\pi}(k,j), P(k,C)$ is positive and monotone increasing. Then using (16) in (14) it is obvious that $\vec{\pi}(k+1,j)$ is also positive. In order to show monotonicity is sufficient to show that $\vec{\pi}(k+1,j) - \vec{\pi}(k+1,j-1)$ is positive for $j = 0 \dots m$.

Case $d_i = 0$. As M is a Markov chain generator, for any positive vector $\vec{\beta}$ the vector $\vec{\beta}e^{Mt}$ is also positive. In this case:

$$\pi_i(k+1,j) - \pi_i(k+1,j-1) = (\vec{\pi}(k,j) - \vec{\pi}(k,j-1)\Theta_i.$$

Then monotonicity of $\vec{\pi}(k,j)$ implies monotonicity of $\pi_i(k+1,j)$.

Case $d > 0$. In this case:

$$\begin{aligned}&\pi_i(k+1,j) - \pi_i(k+1,j-1) = \\ &(1-d_ir)(\vec{\pi}(k,j) - \vec{\pi}(k,j-1))\Theta_i + d_ir(\pi(k,j-1) - \pi(k,j-2))\Theta_i.\end{aligned}$$

Using the same property of the matrix exponential as in the previous case monotonicity of $\vec{\pi}(k,j)$ implies monotonicity of $\pi_i(k+1,j)$. On another hand it is clear that $\pi_i(k,j) > 0$ for $j > 0$. This insures monotonicity in the point 0.

Case $d < 0$. For $j < m$ we have an analogous argument as in the previous case.

For $j = m$ the boundary conditions insures that $\frac{\partial \vec{\pi}(k,x)}{\partial x}$ is positive which implies the monotonicity of $\vec{\pi}(k,j)$ in $j = m$.

The linear system which is to be solved for the boundary conditions is equivalent to the solution of a Markov chain (see the explicit formulas in [6, 4]. It follows that $P(k,C)$ is positive.

As the initial condition is a probability density function the positivity and monotonicity of the solution is assured if (13)(equivalently (16)) holds. $\square$

Note that, positivity, strict monotonicity of $\vec{\pi}$ and the normalisation equation in (16) implies that $\vec{\pi}, \vec{P}(C)$ is a probability distribution function.

4 Conclusion

We have provided a numerical algorithm for the study of the transient behavior of the two unreliable machines tandem system with finite capacity buffer. The numerical complexity of the algorithm looks, at a first view, quite elevated due to the necessity to solve a linear equations system at each step of the algorithm. In the real numerical implementations this is not necessary true as for the case of two state machines the dimension of the system is four and explicit formulas can be deduced in an analogous manner than for the stationary solution.

The proof of the convergence and numerical examples can be retrieved on the author web site: http://www-lag.ensieg.inpg.fr/mocanu

References

[1] D. Anick, D. Mitra, and M. M. Sondhi. Stochastic theory of a datahandling system with multiple sources. *The Bell System Technical Journal*, 61, 1982.

[2] Ph. Ciprut, Hongler M. O., and Y Salama. Fluctuations of the production output of transfer lines. *Journal of Intelligent Manufacturing*, 11:183–189, 2000.

[3] Y. Dallery and S.B. Gershwin. Manufacturing flow line systems: a review of models and analytical results. *Queueing Systems*, 12:3–94, 1992.

[4] D. Dubois and J.-P. Forestier. Productivité et en cours moyen d'un ensemble de deux machines sépartées par une zone de stockage. *RAIRO Automatique*, 16(2):105–132, 1982.

[5] S. B. Gershwin. Representation analysis of transfer lines with machines that have different processing rates. *Annals of Operations Research*, 9:511–530, 1987.

[6] S.B. Gershwin. *Manufacturing Systems Engineering*. Prentice Hall, New Jersey, 1994.

[7] M. O. Hongler. Finite time evolution for the population level of a buffer. Technical Report BIBOS-568-93, BiBoS, Universitat Bielefeld, 1993.

[8] H. Kobayashi and H. Ren. A mathematical theory for transient analysis of communication networks. *IEICE Trans. Comm.*, E75-B(12):1266–1276, 1992.

[9] P. Kulkarni. *Frontiers in Queueing: Models and Applications in Science and Engineering*, chapter Fluid Models for Single Buffer Systems, pages 321–338. CRC Press, 1995.

[10] D. Mitra. Stochastic theory of a fluid model of producers and consumers coupled by a buffer. *Adv. Appl. Prob.*, 20(3):646–676, 1988.

[11] Q. Ren and H. Kobayashi. Transient solutions for the buffer behavior in statistical multiplexing. *Performance Evaluation*, 23(1):65–87, 1995.

[12] B. Sericola. Transient analysis of stochastic fluid models. *Performance Evaluation*, 32(4):245–263, 1998.

[13] B. Sericola and B. Tuffin. A fluid queue driven by a markovian queue. *Queuing Systems – Theory and Applications*, 21(3-4):253–264, 1999.

[14] R. B. Sidje. Expokit: a software package for computing matrix exponentials. *ACM Trans. Math. Softw.*, 24(1):130–156, 1998.

[15] Roger B. Sidje and William J. Stewart. A numerical study of large sparse matrix exponentials arising in Markov chains. *Computational Statistics & Data Analysis*, 29(3):345–368, 1999.

[16] W. J. Stewart. *Introduction to the Numerical Solution of Markov Chains*. Princeton University Press, 1994.

[17] T. Tanaka, O. Hashida, and Y. Takahashi. Transient analysis of fluid model for ATM statistical multiplexer. *erformance Evaluation*, 23(2):145–162, 1995.