ANALYSIS OF FLOW LINES WITH COX-2-DISTRIBUTED PROCESSING TIMES AND LIMITED BUFFER CAPACITY

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Abstract:

We develop a flow line model consisting of machines with Cox-2-distributed processing times and limited buffer capacities. A two-machine subsystem is analyzed exactly and a larger flow lines are evaluated through a decomposition into a set of two-machine lines. Our results are compared to those given by Buzacott, Liu and Shantikumar for their "Stopped Arrival Queue Modell".

Keywords: Flow line, performance evaluation, decomposition, generel processing times, Cox-2-distribution

1 Introduction

We describe an approximate approach to determine the production rate and inventory level of a flow line consisting of more than two machines where adjacent machines are decoupled through buffers of limited capacity. We assume that machines are reliable and that processing times are Cox-2-distributed. This allows us to model processing times with any squared coefficient of variation $c^2 \geq 0.5$. These processing times can include the random delay of workpieces which is due to random failures and repairs of the machines if we use the completion time concept proposed by Gaver (1962).

The paper is structured as follows: In Section 2 we formally describe the type of flow line to be analyzed. Section 3 outlines the exact analysis of the two-machine, one-buffer subsystem that serves as the building block of a decomposition and which has already been analyzed by Buzacott und Kostelski (1987) using the Matrix geometric method. Our analysis of the two-machine system, however, follows the approach for this type of system which is thoroughly explained in Gershwin (1994). The decomposition algorithm is briefly described in Section 4. In Section 5 we present some preliminary numerical results by comparing our results to those obtained from the multistage flow line analysis with the stopped arrival queue model as proposed by Buzacott et al. (1995). Section 6 outlines future research.

2 The Model

We assume that the flow line consists of K machines or stages. The processing times at machine i follow a Cox-2 distribution. Each buffer B_i between machines M_i and M_{i+1} has the capacity to hold up to N_i workpieces that flow from the leftmost to the rightmost machine. An example of such a flow line is depicted in Figure 1.

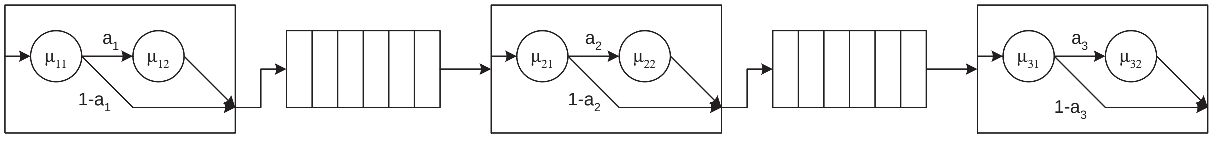

Figure 1: Flow line with three machines

The rates of the two phases of stage i are μ_{i1} and μ_{i2} respectively. The second phase of stage i will be required after completion of phase 1 with probability a_i. Therefore, a workpiece completes its service at stage i with probability $1 - a_i$ after the completion of the first phase and with probability a_i after the completion of the second phase. Note that these states of a machine or stage do not represent servers: No more than one workpiece can be a machine at any moment in time, and if it is there, it is in one out of the two phases of the respective machine. Each machine except for the first and the last can be either idle or blocked or it can be processing in phase one or two. We furthermore assume that the first machine is never starved and the last is never blocked.

3 The Two-Maschine Subsystem

In order to analyze a larger system with more than two machines, we first study a two-machine line. The state of this two-machine line is given by the state of the first machine, the state of the second machine, and the buffer level. In the analysis to follow, we define the buffer level as including all parts that are currently being processed at the second machine, that are waiting in the physical buffer between the first and the second machine, and those parts that have been processed at the first machine but cannot leave it because the physical buffer between the machines is full such that the first machine is blocked. That is, we follow the blockage convention which described in Gershwin (1994, p. 95). In order to describe the state space, we use use the triple (n, α_1, α_2) where n denotes the buffer level, α_1 the state of machine M_1 and α_2 the state of machine M_2.

The probability of the system being in this state is $\mathbf{p}(n, \alpha_1, \alpha_2)$. Machine M_1 can either be in the first phase ($\alpha_1 = 1$), in the second phase ($\alpha_1 = 2$) or it can be blocked ($\alpha_1 = B$). The downstream machine M_2 can either be in the first phase ($\alpha_2 = 1$), in the second phase ($\alpha_2 = 2$) or it can be starved ($\alpha_1 = S$).

This leads to the following transition equations which differ for states with an empty or almost empty buffer, states with a full a almost full buffer, and the intermediate states with a buffer level that is in between:

Lower boundary states:

$$\begin{aligned}
\mu_{11}\mathbf{p}(0,1,I) &= (1-a_2)\mu_{21}\mathbf{p}(1,1,1) + \mu_{22}\mathbf{p}(1,1,2) && (1)\\
\mu_{12}\mathbf{p}(0,2,I) &= a_1\mu_{11}\mathbf{p}(0,1,I) + (1-a_2)\mu_{21}\mathbf{p}(1,2,1) + \\
&\quad \mu_{22}\mathbf{p}(2,1,2) && (2)\\
(\mu_{11}+\mu_{21})\mathbf{p}(1,1,1) &= (1-a_1)\mu_{11}\mathbf{p}(0,1,I) + \mu_{12}\mathbf{p}(0,2,I) + \\
&\quad (1-a_2)\mu_{21}\mathbf{p}(2,1,1) + \mu_{22}\mathbf{p}(2,1,2) && (3)\\
(\mu_{11}+\mu_{22})\mathbf{p}(1,1,2) &= a_2\mu_{21}\mathbf{p}(1,1,1) && (4)
\end{aligned}$$

Intermediate states:

$$\begin{aligned}
(\mu_{11}+\mu_{21})\mathbf{p}(n,1,1) &= (1-a_1)\mu_{11}\mathbf{p}(n-1,1,1) + \mu_{12}\mathbf{p}(n-1,2,1) + \\
&\quad (1-a_2)\mu_{21}\mathbf{p}(n+1,1,1) + \mu_{22}\mathbf{p}(n+1,1,2) \\
&\qquad\qquad (\text{for } 2 \le n \le N-2) && (5)\\
(\mu_{11}+\mu_{22})\mathbf{p}(n,1,2) &= (1-a_1)\mu_{11}\mathbf{p}(n-1,1,2) + \mu_{12}\mathbf{p}(n-1,2,2) + \\
&\quad a_2\mu_{21}\mathbf{p}(n,1,1) \qquad (\text{for } 2 \le n \le N-1) && (6)
\end{aligned}$$

$$
\begin{aligned}
(\mu_{12}+\mu_{21})\mathbf{p}(n,2,1) &= a_1\mu_{11}\mathbf{p}(n,1,1)+\mu_{22}\mathbf{p}(n+1,2,2)+ \\
&\quad (1-a_2)\mu_{21}\mathbf{p}(n+1,2,1) \quad (\text{for } 1\le n\le N-2) && (7)\\
(\mu_{12}+\mu_{22})\mathbf{p}(n,2,2) &= a_1\mu_{11}\mathbf{p}(n,1,2)+a_2\mu_{21}\mathbf{p}(n,2,1) \\
&\qquad\qquad\qquad (\text{for } 2\le n\le N-1) && (8)
\end{aligned}
$$

Upper boundary states:

$$
\begin{aligned}
\mu_{21}\mathbf{p}(N,B,1) &= (1-a_1)\mu_{11}\mathbf{p}(N-1,1,1)+ \\
&\quad \mu_{12}\mathbf{p}(N-1,2,1) && (9)\\
\mu_{22}\mathbf{p}(N,B,2) &= a_2\mu_{21}\mathbf{p}(N,B,1)+ \\
&\quad (1-a_1)\mu_{11}\mathbf{p}(N-1,1,2)+ \\
&\quad \mu_{12}\mathbf{p}(N-1,2,2) && (10)\\
(\mu_{11}+\mu_{21})\mathbf{p}(N-1,1,1) &= (1-a_1)\mu_{11}\mathbf{p}(N-2,1,1)+\mu_{12}\mathbf{p}(N-2,2,1)+ \\
&\quad (1-a_2)\mu_{21}\mathbf{p}(N,2,1)+\mu_{22}\mathbf{p}(N,2,2) && (11)\\
(\mu_{12}+\mu_{21})\mathbf{p}(N-1,2,1) &= a_1\mu_{11}\mathbf{p}(N-1,1,1) && (12)
\end{aligned}
$$

Together with the normalization equation stating that all probabilities add up to one this leads to a linear system of equations which can be solved in several ways. An almost identical system of equations has been formulated by Buzacott und Kostelski (1987) and solved via the matrix geometric method and a recursive algorithm. Since the methods suffered from numerical instabilities, we developed a solution techniques using the ideas for the analysis of two-machine models presented in Gershwin (1994, pp.105). This leads to a numerically stable algorithm providing the exact values of all the system states as well as the performance measures such as the production rate and the inventory level.

4 The Decomposition Approach

While it is possible to analyze a two-machine system exactly, the exact analysis of larger systems is practically impossible as the state space of the system explodes very quickly. For this reason decomposition approaches are frequently used to analyze larger systems. The basic idea is to decompose a system with K machines and $K-1$ buffers into $K-1$ two-machine systems with virtual machines that mimic to an observer in the buffer the flow of material in and out as it would be seen in the corresponding buffer of the real system. We followed the ideas presented in great detail in Gershwin (1994) to develop an iterative decomposition algorithm to analyze flow lines with more than two machines. However, some modifications were necessary which we will now briefly outline. While the models analyzed in Gershwin (1994) assumed unreliable machines and consequently lead to so-called *interruption-of-flow-* and *resumption-of-flow*-equations, we are studying a flow line with reliable machines which cannot fail. The machines in our system, however, change their phases of operation as described in 2. For this reason, we derived the following three types of decomposition equations:

- **Phase-One-to-Two (P1t2)-Equation:** This equation deals with the probability of the transition of the virtual machine from its first phase of operation to its second.
- **Phase-Two-to-One (P2t1)-Equation:** This equation deals with the probability of the transition of the virtual machine from its second phase of operation to its first.
- **Flow-Rate-Idle-Time (FRIT)-Equation:** This is a type of equation which relates the flow of material through a machine to the machine's isolated production rate and its probability of being blocked or starved. This type of equation has also been used by Gershwin et al.

In the following we will briefly discuss the derivation of the parameters of the virtual machines. We will concentrate on the P1t2- and the P2t1-equations since the FRIT-equation is rather similar to those derived for other flow line models.

Central to the derivation of the P1t2- and P2t1-equations is the definition of virtual machine states. We study a line $L(i)$ which is related to the buffer between machines M_i and M_{i+1}. The virtual machines of line $L(i)$ are $M_u(i)$ (upstream of the buffer) and $M_d(i)$ (downstream of the buffer). We want to determine the parameters $a_u(i)$, $\mu_{u_1}(i)$, and $\mu_{u_2}(i)$ of the virtual machine $M_u(i)$ as well as the parameters $a_d(i)$, $\mu_{d_1}(i)$, and $\mu_{d_2}(i)$ of the virtual machine $M_d(i)$ in order to be able to use our two-machine model from Section 3 to determine performance measures for the flow line.

We assume that the virtual machine $M_u(i)$ is in phase one if the real machine M_i is processing a workpiece or when it is *waiting* (starved), $\alpha_i(t) = S$) for the next workpiece:

$$\begin{aligned} \{\alpha_u(i,t) = 1\} \quad \text{iff} \quad & \{\alpha_i(t) = 1\} \text{ and } \{n(i,t) > 0\} \text{ or} \\ & \{\alpha_i(t) = S\} \end{aligned} \tag{13}$$

If the real machine M_i is in phase two, so is machine $M_u(i)$::

$$\{\alpha_u(i,t) = 2\} \quad \text{iff} \quad \{\alpha_i(t) = 2\} \tag{14}$$

To derive the P1t2-equation, we ask for the probability of observing a transition from phase one to phase two. For this to happen, we have to observe a completion of phase one (with probability $\mu_u(i)\delta t$ *and* the process must enter the second phase (which happens with probability $a_u(i)$). The joint probability can be related to a change in the machine states defined above:

$$a_u(i)\mu_{u_1}(i)\delta t = \text{Prob}[\{\alpha_u(i,t+\delta t) = 2\}|\{\alpha_u(i,t) = 1\}] \tag{15}$$

If we insert the definition of the virtual machine states given in (13) and (14), we get the following result:

$$a_u(i)\mu_{u_1}(i)\delta t = a_i\mu_{i1}\delta t\text{Prob}[\{n(i-1,t) > 0\}] \tag{16}$$

A transition of the virtual machine from state two to state one, however, can only occur if the real machine M_i completes the second phase of operation on a workpiece:

$$\mu_{u_2}(i)\delta t = \text{Prob}[\{\alpha_u(i,t+\delta t) = 1\}|\{\alpha_u(i,t) = 2\}] \tag{17}$$

If we insert the definition of the virtual machine states given in (13) and (14), we get the following result:

$$\mu_{u_2}(i)\delta t = \mu_{i2}\delta t \tag{18}$$

The FRIT-equation (which is not described here) is the third equation that is needed to determine the three parameters $a_u(i)$, $\mu_{u_1}(i)$, and $\mu_{u_2}(i)$ of the virtual machine $M_u(i)$. The approach for the downstream machine is analogous. We then reformulated and solved the decomposition equations using a method similar to the one given in Burman (1995, p. 84-87) and Helber (1999, p. 71-76).

5 Numerical Results and Conclusion

In order to evaluate the accuracy of the algorithm, we compared it to results given in Buzacott et al. (1995). In these cases, the expected value $\frac{1}{\mu_i}$ and the coefficient of variation c_i^2 of the processing time for each machine M_i was given. The Cox-2-distribution, however, has three parameters, so that one degree of freedom is left. We used the so-called "balanced-mean" two-phase Coxian distribution given in Buzacott und Shanthikumar (1993, p. 450-451) to match the problem data in Buzacott et al. (1995). Table 1 gives parameters and production rate estimates for a three-stage system with general service times and two buffer spaces between adjacent machines. We report results for the cases 2, 3, 4, and 7 from Table 5 in in Buzacott et al. (1995). These are cases with a squared coefficient of variation $c_i^2 \geq 0.5$ for which our algorithm is applicable.

Table 1: Three-stage system with general service times

Case	2	3	4	7
μ_1	0.5	0.5	0.5	0.5
μ_2	0.5	0.5	0.5	1.0
μ_3	0.5	0.5	0.5	0.5
c_1^2	0.5	0.8	2.0	0.6
c_2^2	0.5	0.8	2.0	0.6
c_3^2	0.5	0.8	2.0	0.6
Sim. PR	0.382	0.351	0.296	0.427
BLS-a (abs.)	0.384	0.347	0.272	0.441
BLS-a (rel.)	0.52%	-1.14%	-8.11%	3.28%
BLS-b (abs.)	0.381	0.349	0.282	0.429
BLS-b (rel.)	-0.26%	-0.57%	-4.73%	0.47%
CoxDC (abs.)	0.380	0.348	0.294	0.444
CoxDC (rel.)	-0.52%	-0.85%	-0.68%	3.98%

Table 2: Four-stage system with exponential service times

Case	1	2	3	4
μ_1	1.0	1.0	1.0	1.0
μ_2	1.1	1.2	1.5	2.0
μ_3	1.2	1.4	2.0	3.0
μ_4	1.3	1.6	2.5	4.0
Exact. PR	0.71	0.765	0.861	0.929
BLS-a (abs.)	0.689	0.746	0.85	0.925
BLS-a (rel.)	-2.96%	-2.48%	-1.28%	-0.43%
BLS-b (abs.)	0.7	0.756	0.855	0.927
BLS-b (rel.)	-1.41%	-1.18%	-0.70%	-0.22%
CoxDC (abs.)	0.705	0.762	0.86	0.929
CoxDC (rel.)	-0.70%	-0.39%	-0.12%	0.00%

The entries "BLS-a" and "BLS-b" are related to two approximation techniques described in Buzacott et al. (1995), "Sim" denotes the simulation results and "CoxDC" the results from our approach. For the cases in Table 1 our approach yields results of comparable accuracy.

For the four-stage systems in Table 2 with exponential service times our approach gives more accurate results that the procedures proposed in Buzacott et al. (1995).

Table 3: Three-stage system with general service times

Case	1	2	3
μ_1	0.5	0.5	0.5
μ_2	0.5	0.5	0.5
μ_3	0.5	0.5	0.5
c_1^2	0.75	2.0	2.0
c_2^2	0.75	2.0	2.0
c_3^2	0.75	2.0	2.0
Sim. PR	0.385	0.322	0.360
BLS-a (abs.)	0.385	0.303	0.345
BLS-a (rel.)	0.00%	-5.90%	-4.17%
BLS-b (abs.)	0.385	0.312	0.349
BLS-b (rel.)	-0.00%	-3.11%	-3.06%
Altiok (abs.)	0.368	0.338	0.368
Altiok (rel.)	-4.42%	4.97%	2.22%
CoxDC (abs.)	0.385	0.322	0.359
CoxDC (rel.)	0.00%	0.00%	-0.28%

Table 3 presents results for systems given by Altiok as reported in Buzacott et al. (1995). In all cases there are three buffer spaces between adjacent machines except for Case 3 with 10 buffer spaces between machines 2 and 3. For these systems our approach outperforms the other methods.

Given the numerical results we conclude that our decomposition approach can be used to analyze flow lines with general service times as long as these service times have a squared coefficient of variation larger than 0.5.

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